On the inductive representation of many-dimensional recursively enumerable sets definable in some arithmetical structures

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ABSTRACT

Algebras Ω^0, Ω_1, Ω_2, Ω_3 of many-dimensional recursively enumerable fuzzy sets (REFS) based on the operations +(sum), •(product), ×(Cartezian product), $\downarrow_i$ (projection on x_i), Tij (transposition of x_i and x_j) are introduced as well as algebras θ^0, θ_1, θ_2, θ_3 of many-dimensional recursively enumerable sets (RES) in the usual sense based on similar operations $\cup$, $\cap$, $\tilde{\times}$, $\tilde{\downarrow}_i$, $\tilde{T}_{ij}$. Arithmetical structures N_A= (N, =, S, +, 0), N_L= (N, =, <, S, 0), N_S= (N, =, S, 0) on the set N= { 0, 1, 2, ...} of natural numbers, where S(x)= x+1, are considered. It is proved that any REFS is inductively representable in Ω^0 up to the equivalence (correspondingly, in Ω_1, Ω_2) if and only if it is definable in N_A (correspondingly, in N_L, N_S). It is proved also that any RES is inductively representable in θ^0 (correspondingly, in θ_1, θ_2) if and only if it is definable in N_A (correspondingly, in N_L, N_S). Theorems are proved concerning the inductive representability of REFS in Ω_3 and RES in θ_3.

Keywords

Fuzzy set, recursively enumerable set, predicate formula, signature, structure.

In this report many-dimensional recursively enumerable fuzzy sets (REFS) as well as many-dimensional recursively enumerable sets (RES) in the usual sense are considered. Some theorems are given concerning the inductive representation of the sets of mentioned kinds definable in the arithmetical structures considered below. These results are actually generalizations of some theorems given in [9], [11], [12].

Let us recall some definitions connected with RESes and REFSes (cf. [6], [9], [11], [12]).n-dimensional RES is defined as recursively enumerable set of n-tuples ($x_1, x_2, ..., x_n$) where $x_i \in$ N={0, 1, 2, ...} for $1\le i \le n$. The operations of <u>union</u> $\cup$ and <u>intersection</u> $\cap$ of n-dimensional RESes are defined in an usual way. <u>Cartezian product</u> A $\tilde{\times}$ B of RESes A and B having the dimensions, correspondingly, n and m, is defined by the following generating rule (g.r.) : if ($x_1, x_2, ..., x_n$) $\in$ A and ($y_1, y_2, ..., y_m$) $\in$ B then ($x_1, x_2, ..., x_n, y_1, y_2, ..., y_m$) $\in$ A $\tilde{\times}$ B. <u>Projection</u> $\tilde{\downarrow}_i$ A on x_i (where $1 \le i \le n$) is defined by the following g.r. : if ($x_1, x_2, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n$) $\in$ A then ($x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n$) $\in$ $\tilde{\downarrow}_i$ A. <u>Transposition</u> $\tilde{T}_{ij}$ A of x_i and x_j in an n-dimensional RES A (where $1 \le i, j \le n$) is defined by the following g.r. : if ($x_1, ..., x_i, ..., x_j, ..., x_n$) $\in$ A then ($x_1, ..., x_j, ..., x_i, ..., x_n$) $\in$ $\tilde{T}_{ij}$ A. The RESes $\tilde{Z}_0$, $\tilde{R}$, $\widetilde{\mathrm{Add}}$, $\tilde{Q}$, $\tilde{J}$ having the dimensions, correspondingly 1, 2, 3, 2, 2 are defined as follows: $\tilde{Z}_0$ = {x | x = 0}; $\tilde{R}$ = { (x, y) | y = x + 1}; $\widetilde{\mathrm{Add}}$ = {(x, y, z) | z = x + y}; $\tilde{Q}$ = {(x, y) | x < y}; $\tilde{J}$ = { (x, y) | x $\ne$ y }.

Let us consider some algebras on the set of all many-dimensional RESes. The notion of algebra is interpreted as "universal algebra" ([2], [5], see also [9], [11], [12]). The algebras θ^0, θ_1, θ_2, θ_3 are defined by the list of operations ($\cup$, $\cap$, $\tilde{\times}$, $\tilde{\downarrow}_i$, $\tilde{T}_{ij}$) and by the following lists of basic elements : ($\tilde{Z}_0$, $\tilde{R}$, $\widetilde{\mathrm{Add}}$) for θ^0 ; ($\tilde{Z}_0$, $\tilde{R}$, $\tilde{Q}$) for θ_1 ; ($\tilde{Z}_0$, $\tilde{R}$, $\tilde{J}$) for θ_2 ; ($\tilde{Z}_0$, $\tilde{R}$) for θ_3. Note, that these algebras are different from the algebras having the same notations in [9], [11], [12]. The relations between the mentioned algebras will be considered below. We say that an element belonging to the domain of algebra is <u>inductively representable</u> in it if this element can be constructed from the basic elements of the considered algebra using the operations of the algebra.

Note that in [6] it is proved (see [6] , lemma 1), that every many-dimensional RES can be obtained from $\tilde{Z}_0$ and $\tilde{R}$ using the operations $\cup$, $\cap$, $\tilde{\times}$, $\tilde{\downarrow}_i$, $\tilde{T}_{ij}$ and the operation of transitive closure. Similar statement is actually proved in [14] concerning the inductive representation of RESes consisting of n-tuples of words in a fixed alphabet.

The n-dimensional recursively enumerable fuzzy set (REFS) is defined as a recursively enumerable set of (n + 1)-tuples ($x_1, x_2, ..., x_n$, ε), where $x_i \in$ N for $1\le i \le n$ and ε is a binary rational number $\frac{k}{2^m}$, such that $0\le \frac{k}{2^m} \le 1$. We consider the following operations on REFSes. The <u>sum</u> W+U of n-dimensional REFSes W and U is defined by the following g.r. : if ($x_1, x_2, ..., x_n, \varepsilon$) $\in$ W and ($x_1, x_2, ..., x_n, \delta$) $\in$ U, then ($x_1, x_2, ..., x_n$, min(1, $\varepsilon + \delta$)) $\in$ W+U. The product W•U of n-dimensional REFSes W and U is defined by the following g.r. : if ($x_1, x_2, ..., x_n, \varepsilon$) $\in$ W and ($x_1, x_2, ..., x_n, \delta$) $\in$ U, then ($x_1, x_2, ..., x_n, \varepsilon \bullet\delta$)) $\in$ W•U. The Cartesian product W×U of n-dimensional REFS W and m-dimensional REFS U is defined by the following g.r. : if ($x_1, x_2, ..., x_n, \varepsilon$) $\in$ W and ($y_1, y_2, ..., y_m, \delta$) $\in$ U then ($x_1, x_2, ..., x_n, y_1, y_2, ..., y_m, \varepsilon \bullet\delta$) $\in$ W×U. The projection $\downarrow_i$ W of n-dimensional REFS W on x_i (where $1 \le i \le n$) is defined by the following g.r. : if ($x_1, x_2, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n, \varepsilon$) $\in$ W then ($x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n, \varepsilon$) $\in$ $\downarrow_i$ W. The transposition T_{ij}W of x_i and x_j in W (where $1 \le i, j \le n$) is defined by the following g.r. : if ($x_1, ..., x_i, ..., x_j, ..., x_n, \varepsilon$) $\in$ W then ($x_1, ..., x_j, ..., x_i, ..., x_n, \varepsilon$) $\in$ T_{ij}W. The REFSes Z_0, R, Q, Add, J, H having the dimensions, correspondingly, 1, 2, 2, 3, 2,1 are defined by the following g.r. : (x, 0) $\in$ Z_0 for any $x \in$ N; (0, 1) $\in$ Z_0; (x, y, 0) $\in$ R for any $x \in$ N, $y \in$ N; (x, x +1, 1) $\in$ R for any $x \in$ N; (x, y, 0) $\in$ Q for any $x \in$ N, $y \in$ N; (x, y, 1) $\in$ Q if and only if $x \in$ N, $y \in$ N, $x < y$; (x, y, z, 0) $\in$ Add for any $x \in$ N, $y \in$ N, $z \in$ N; (x, y, z, 1) $\in$ Add if and only if $x \in$ N, $y \in$ N, $z \in$ N, $x + y = z$; (x, y, 0) $\in$ J for any x $\in$ N, y $\in$ N; (x,y,1) $\in$ J if and only if x$\in$ N, y $\in$ N, $x \neq y$; (x, $\frac{1}{2}$) $\in$ H and (x, 0) $\in$ H for any $x \in$ N.

We say that n-dimensional REFSes W and U are equivalent if for any (n+1)-tuple ($x_1, x_2, ..., x_n, \varepsilon$) $\in$ W, where $\varepsilon > 0$, there exists an (n+1)-tuple ($x_1, x_2, ..., x_n, \delta$) $\in$ U such that $\delta \ge \varepsilon$, and also for any (n+1)-tuple ($x_1, x_2, ..., x_n, \varepsilon$) $\in$ U, where $\varepsilon > 0$, there exists an (n+1)-tuple ($x_1, x_2, ..., x_n, \delta$) $\in$ W such that $\delta \ge \varepsilon$.

Such notion of equivalence is considered in [6], [8], [9], [11], [12]. An n-dimensional REFS W is said to be m-discrete for some natural m if for any (n+1)-tuple ($x_1, x_2, ..., x_n, \varepsilon$) $\in$ W there exists such k that $0 \le k \le 2^m$ and $\varepsilon = \frac{k}{2^m}$. An n-dimensional REFS W is said to be discrete if it is m-discrete for some m. For every m-discrete n-dimensional REFS W its ε_0-level; W[ε_0], where ε_0 is a binary rational number such that $0 \le \varepsilon_0 \le 1$, is defined as the RES of n-tuples ($x_1, x_2, ..., x_n$) such that ($x_1, x_2, ..., x_n, \varepsilon_0$) $\in$ W. We shall say in such cases that ε_0 is the index of ε_0-level W [ε_0] in the REFS W.

We consider the algebras Ω^0, Ω_1, Ω_2, Ω_3 on the set of all REFSes defined by the operations +, •, ×, $\downarrow_i$, T_{ij} and the following lists of basic elements: (Z_0, R, Add, H) for Ω^0, (Z_0, R, Q , H) for Ω_1, (Z_0, R, J, H) for Ω_2, (Z_0,R, H) for Ω_3. Note that these algebras are different from algebras having the same notations in [9], [11], [12]. The relations between the mentioned algebras will be considered below.

Note that in [6] it is proved that any REFS can be constructed from Z_0, R, H up to the equivalence using the operations +, •, ×, $\downarrow_i$, T_{ij} and the operations of additive-transitive closure and multiplicative-transitive closure.

The notion of predicate formula is defined as in [1] and [4] (see also [9], [11], [12]). Signature is defined as any set of predicate symbols, functional symbols and symbols of constants. The notion of structure in a given signature is defined as in [1], namely, a structure in a given signature Σ is a system consisting of some non-empty set M (the universe of the structure) and an assignment which assigns to each n-dimensional predicate symbol (correspondingly, n-dimensional functional symbol) belonging to Σ an n-dimensional predicate (correspondingly n-dimensional function) on M and assigns to each symbol of constant belonging to Σ an element of the universe M. The notion of truth of a given predicate formula F in a signature Σ concerning a structure T in Σ for given values of the free variables $x_1, x_2, ..., x_n$ in F is defined in a natural way (see [1]). Let us consider (cf.[1]) the following structures on the set N={0, 1, 2, ...} where **S** is the function S(x)=x+1 and the notations =, <, +, 0 are interpreted in a natural way :

(1) $\mathbf{N}_A$ is the structure (N, =, **S**, +, 0).
(2) $\mathbf{N}_L$ is the structure (N, =, <, **S**, 0).
(3) $\mathbf{N}_S$ is the structure (N, =, **S**, 0).

Note, that these structures are considered in [1]. The structure $\mathbf{N}_A$ is described by the system of formal arithmetic introduced by M. Presburger (see [13]).

We say that an n-dimensional RES A is expressed by the formula F in the signature of a structure T on the set N if for any values $k_1 \in$ N, $k_2 \in$ N, ..., $k_n \in$ N of free variables $x_1, x_2, ..., x_n$ in F the following condition holds: the formula F is true concerning T for $x_1 = k_1$, $x_2 = k_2$, ..., $x_n = k_n$ if and only if ($k_1, k_2, ..., k_n$)$\in$ A. We say that n-dimensional RES A is definable in a structure T if there exists a formula F in the signature of T such that A is expressed by F. We say that an n-dimensional REFS W is definable in a structure T if it is discrete and all its ε_0-levels W [ε_0] are RESes definable in T.

A formula F in the signature of the structure $\mathbf{N}_S$ is said to be positive if it contains no other logical symbols except $\exists$, &, $\vee$, $\neg$ (so, it does not contain $\forall$, $\supset$, ~), and all the negation symbols in F relate only to the elementary subformulas (t =s) containing no more than one variable (cf. [12]).

Theorem 1. A many-dimensional RES is inductively representable in the algebra θ^0 (correspondingly, in the algebra θ_1 or in the algebraθ_2) if and only if it is definable in the structure $\mathbf{N}_A$ (correspondingly, in the structure $\mathbf{N}_L$ or in the structure $\mathbf{N}_S$).

Theorem 2. A many-dimensional RES is inductively representable in the algebra θ_3 if and only if it can be expressed by a positive formula in the structure $\mathbf{N}_S$.

Theorem 3. A many-dimensional REFS W is inductively representable in the algebra Ω^0 up to the equivalence (correspondingly, in the algebra Ω_1 or in the algebra Ω_2) if and only if it is definable in the structure $\mathbf{N}_A$ (correspondingly, in the structure $\mathbf{N}_L$ or in the structure $\mathbf{N}_S$).

Theorem 4. If a many-dimensional REFS is inductively representable in the algebra Ω_3 then it is discrete and all its ε_0-levels can be expressed by positive formulas in the structure $\mathbf{N}_S$.

The question whether the statement inverse to the Theorem 4 is true or not, remains open.

Corollary. A RES is inductively representable in the algebra denoted by θ^0 in [9] and [11] (correspondingly, θ_1, θ_2, θ_3 in [12]) if and only if it is two-dimensional and is inductively representable in the algebra denoted by the same notation in this report. A RFES is inductively representable in the algebra denoted by Ω^0 in [9] and [11] (correspondingly, Ω_1, Ω_2 in [12]) if and only if it is two-dimensional and is inductively represented in the algebra denoted by the same notation in this report.

An analogous question concerning Ω_3 remains open.

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