On Upper Cone of T-degrees Containing wtt-mitotic but not tt-mitotic Sets

Arsen H. Mokatsian

Institute for Informatics and Automation Problems

National Academy of Sciences of Armenia

e-mail: yuri@arminco.com

ABSTRACT

In this article it is shown, the existence of a low computably enumerable (c.e.) T-degree $\mathbf{u}$ such that if $\mathbf{v}$ is a c.e. T-degree and $\mathbf{u} \le \mathbf{v}$ then $\mathbf{v}$ contains wtt-mitotic but non-tt-mitotic set.

We shall use notions and terminology introduced in [5], [6]. The definitions of tt- and wtt- reducibilities are from [5].

Definition. The order pair $\ll x_1,\cdots,x_k >, \alpha >$, where $< x_1,\cdots,x_k >$ is a k-tuple of integers and α is a k-ary Boolean function ($k>0$) is called a *truth-table condition* (or *tt-condition*) *of norm* k. The set $\{x_1,\cdots,x_k\}$ is called the *associated set of the tt-condition*.

Definition. The tt-condition $\ll x_1,\cdots,x_k >, \alpha >$, is *satisfied* by A if $\alpha(c_A(x_1),\cdots,c_A(x_k))=1$, where c_A is characteristic function for A.

Each tt-condition is a finite object; clearly an effective coding can be chosen which maps all tt-conditions (of varying norm) onto ω. Assume henceforth that a particular such coding has been chosen. Where we speak of "tt-condition x", we shall mean the tt-condition with the code number x.

Definition. A is *truth-table reducible to* B (notation: $A \le_{tt} B$) if there is a computable function f such that for all x, [$x \in A \Leftrightarrow tt$-condition $f(x)$ is satisfied by B]. We also abbreviate "truth-table reducibility" as "tt-reducibility".

Definition. A is weak truth-table reducible to B (notation: $A \le_{wtt} B$) if $(\exists z)[\, c_A = \varphi_z^B$ ($\exists$ and computable f) $(\forall x)[\, D_{f(x)}$ contains all integers whose membership in B is used in the computation of $\varphi_z^B(x)$]].

Definition. A c.e. set is tt- mitotic (wtt- mitotic) set if it is the disjoint union of two c.e. sets both of the same tt-degree (wtt-degree) of unsolvability.

Let us modify notations defined in [4] with the purpose to adapt them to our theorem.

Let $A \le_{tt} B$ and $(\forall x)[\, x \in A \Leftrightarrow tt$-condition $f(x)$ is satisfied by B] and $\varphi_n = f$. Then we say that $A \le_{tt} B$ *by* φ_n.

Define $(A_0, A_1, \varphi_0, \varphi_1)$ *is* $[c.e.\ tt-split;\ A \le_{tt}]$ if the following hold:
A_0 and A_1 are c.e., $A_0 \cup A_1 = A$, $A_0 \cap A_1 = \varnothing$, $A \le_{tt} A_0$ by ψ_0 and $A \le_{tt} A_1$ by ψ_1.

Let h be a recursive function from ω onto ω^4.
Define $(Y_i, Z_i, \vartheta_i, \psi_i)$ to be a quadruple $(W_{i_o}, W_{i_1}, \varphi_{i_2}, \varphi_{i_3})$, where $h(i)=(i_0,i_1,i_2,i_3)$. If A is c.e. then we say that the *non-tt-mitotic condition of i order is satisfied for A*, if it is not the case $(Y_i, Z_i, \vartheta_i, \psi_i)$ is $[c.e.\ tt-split;\ A \le_{tt}]$.

Definition. $(Y_i, Z_i, \vartheta_i, \psi_i)$ is *threatening A through x at stage s*, if following hold:

(i) $i \le s$,

(ii) $(\forall x_{\le y})\left(\vartheta_i^s(y)\downarrow \& \psi_i^s(y)\downarrow\right)$ and
$\alpha_i(c_{Z_i}(x_1),\cdots,c_{Z_i}(x_{k_i}))=0$ and
$\beta_i(c_{Z_i}(y_1),\cdots,c_{Z_i}(y_{n_i}))=0$,

(iii) $Y_i^s \cap Z_i^s = \varnothing$,

(iv) $A^s(m) = (Y_i^s \cap Z_i^s)(m)$ for all $m \le \max\{s, x_{k_i}, y_{n_i}\}$,

where $\vartheta_i(y) = \ll x_1,\cdots,x_{k_i} >, \alpha_i >$,

$\psi_i(y) = \ll y_1,\cdots,y_{n_i} >, \beta_i >$.

In [3] it is proved that there is a low c.e. T-degree $\mathbf{u}$ such that if $\mathbf{v}$ is a c.e. T-degree and $\mathbf{u} \le \mathbf{v}$ then $\mathbf{v}$ is not completely mitotic.

Theorem. There exists a low c.e. T-degree $\mathbf{u}$ such that if $\mathbf{v}$ is a c.e. T-degree and $\mathbf{u} \le \mathbf{v}$ then $\mathbf{v}$ contains wtt-mitotic but non-tt-mitotic set.

Proof. This statement is proved using a finite injury priority argument. We construct a member U of $\mathbf{u}$ in stages s, $U = \bigcup_{s\in\omega} U_s$. We also construct sets $\{V_e\}_{e\in\omega}$ to witness that each c.e. T-degree in upper cone of $\mathbf{u}$ contains a wtt-mitotic but non-tt-mitotic set.

Denote $\omega^0 = \{x : \exists y\,(2y)=x\}$, $\omega^1 = \omega \setminus \omega^0$.

Construct U, $\{V_e\}_{e\in\omega}$ to satisfy, for all $e\in\omega$, the requirements:

N_e: $\{e\}^U(e)\downarrow$ has a limit in S, the stage.

P_e: $W_e=\Lambda^{V_e}$ for some computable functional Λ.

$R_{\langle e,i\rangle}$: The non-tt-mitotic condition of order i is satisfied for V_e.

We also ensure by permitting that $V_e\equiv_T U\oplus W_e$ and else $V_e^0\equiv_{wtt}V_e^1$ (where $V_e^0=V_e\cap\omega^0$ & $V_e^1=V_e\cap\omega^1$).

If $U\le_T W_e$ then the above ensure that $V_e\equiv_T U\oplus W_e\equiv_T W_e$ and V_e is not *tt*-mitotic. Hence, $deg(W_e)$ is not *tt*-mitotic but is wtt-mitotic, and $\mathbf{u}=deg(U)$ is the required degree.

Let $\langle\ ,\ \rangle$ be computable bijective pairing function increasing in both coordinates. At each stage S place markets $\lambda(e,x,s)$ on elements of $\overline{V}_{e,s}$. Values of λ will be used both as witnesses to prevent the *tt*-mitoticity of V_e sets (by corresponding $Y_i,Z_i,\vartheta_i,\psi_i$) and to ensure that W_e is T-reducible to V_e. Initially $\lambda(e,x,0)=4(\langle e,x\rangle+1)-2)$ for all $e,x\in\omega$.

Also define a function $\xi(e,i,s)$ for all $e,i\in\omega$ (at each stage s), $\xi(e,i,0)=i$ for all $e,i\in\omega$. We use ξ to ensure that only members of sufficiently large magnitude enter U at stage s, so we can satisfy the lowness requirements N_e.

For all $i<j$ the requirement N_i takes priority of the R_i and, naturally, $N_i(R_i)$ takes priority of the $N_j(R_j)$.

The $\{P_e\}_{e\in\omega}$ do not appear in this ranking.

N_e *requires attention* if it is not *satisfied* and $\{e\}^U(e)[s]\downarrow$.

$R_{\langle e,i\rangle}$ *requires attention* if it is not *satisfied* and $(\forall x_{\le y})\left(\vartheta_i^s(y)\downarrow\&\psi_i^s(y)\downarrow\right)$, where $y=\lambda(e,\xi(e,i,s)s)$.

We will build $U=\bigcup_s U_s$ and $V_s=\bigcup_s V_{e,s}$ for all $e\in\omega$. Initially all requirements N_e, $R_{\langle e,i\rangle}$ are declared *unsatisfied.*

Construction.

Stage $s=0$. Let $U_0=\varnothing$, $V_{e,0}=\varnothing$ for all $e\in\omega$.

Stage $s+1$. **Part A** Act on the highest priority requirement which requires attention, if such a requirement exists.

a) If N_e *requires attention* then set $\xi\left(\hat e,\hat i,s+1\right)=\xi\left(\hat e,\hat i+s,s\right)$ for each $\langle\hat e,\hat i\rangle\ge e$. This action prevents injury to N_e by lower priority requirements as we assume that s bounds the use of the halting computation.
Declare N_e satisfied; declare all lower priority R, N unsatisfied.

If $R_{\langle e,i\rangle}$ doesn't require attention, then define $\xi(\ ,\ ,s+1)$ not specified and $\lambda^*(\ ,\ ,s+1)$, $V^*_{e,s+1},U_{s+1}$ to be the same as $\xi(\ ,\ ,s)$, $\lambda(\ ,\ ,s)$, $V_{e,s}$, U_s respectively and go to Part B.

If $R_{\langle e,i\rangle}$ require attention via $y=\lambda\left(e,\xi(e,i,s),s\right)$ then set $\tilde V_{e,s+1}=V_{e,s}\cup\{y-1,y-2\}$ and $\tilde U_{s+1}=U_s\cup\{y-1\}$

b) If ($Y_i,Z_i,\vartheta_i,\psi_i$) is threatening $\tilde V_{e,s+1}$ through y at stage $s+1$ then set $V^*_{e,s+1}=\tilde V_{e,s+1}\cup\{y\}$ and $U_{s+1}=\tilde U_{s+1}\cup\{y\}$.
Whether ($Y_i,Z_i,\vartheta_i,\psi_i$) is threatening $\tilde V_{e,s+1}$ through y at stage $s+1$ or not define $\lambda^*(e,\xi(e,i,s),s+1)=\lambda\left(e,\xi(e,i+\hat s,s),s\right)$,
where $\vartheta_i(y)=$ **<<** $x_1,\cdots,x_{k_i}$ **>**, α_i **>**,
$\psi_i(y)=$ **<<** $y_1,\cdots,y_{n_i}$ **>**, β_i **>**,
$\hat s=\max\{s,x_{k_i},x_{n_i}\}$.

Such definition of λ^* allow us to satisfy $R_{\langle e,i\rangle}$ requirement (after Part A) whether ($Y_i,Z_i,\vartheta_i,\psi_i$) is threatening $\tilde V_{e,s+1}$ through y or not (if don't take into consideration higher priority requirements).
Declare $R_{\langle e,i\rangle}$ satisfied; declare all lower priority R, N unsatisfied.

Part B. If $x\in W_{e,s+1}\setminus W_{e,s}$ then set
$$V_{e,s+1}=V^*_{e,s+1}\cup\left\{\lambda^*(e,x,s+1)\right\}\quad\text{and}$$
$\lambda(e,x+j,s+1)=\lambda^*(e,\xi(e,x+j+1,s+1),s+1)$ for all j∈ ω.

Find all $\hat i$ such that $\lambda(e,\xi(e,\hat i,s+1),s)\ge\lambda^*(e,x,s+1)$ set and declare $R_{\langle e,\hat i\rangle}$ *unsatisfied* for each such $\hat i$.

Define $\lambda(\ ,\ ,s+1)$ not specified in part B above to be the same as $\lambda^*(\ ,\ ,s+1)$.

Note that for all $s,\xi(e,i,s)$ is increasing in both e and i.

Verification

Lemma 1 *For all* e, i*:*

1. N_e is met.
2. $\lim_s\xi(e,i,s)=\xi(e,i)$ exists.
3. $R_{\langle e,i\rangle}$ is met.
4. $\lim_s\lambda\left(e,\xi(e,i,s),s\right)$ exists.

Proof. (1) and (2). The proof is similar to Lemma 1 of Theorem 2.2.2 [3].

(3) and (4). By induction on $j=\langle e,i\rangle$.

Suppose there exists a stage s_0 such that for all $\hat e,\hat i$ with $\left\langle\hat e,\hat i\right\rangle<j$:

3. $R_{\langle\hat e,\hat i\rangle}$ is met and never acts after stage s_0.
4. $\lim_s\lambda\left(\hat e,\xi(\hat e,\hat i,s),s\right)$ exists and is attained by stage s_0.

Then (3) and (4): After s_1 when $W_{e,s}\restriction\ \xi(e,i)+1=W_e\restriction\ \xi(e,i)+1$, then only $R_{\langle e,i\rangle}$ can move $\lambda\left(e,\xi(e,i),s\right)$, $R_{\langle e,i\rangle}$ then acts at most once and is met, say by stage $s_2>s_1$ (when ($Y_i,Z_i,\vartheta_i,\psi_i$) is threatening $\tilde V_{e,s_2}$ through y) because of definition of λ^* (at item a) of Part A) and the fact, that $\lambda\left(e,\xi(e,i),s_2\right)=\lim_s\lambda\left(e,\xi(e,i,s),s\right)$.

Lemma 2 $V_e \leq_T U \oplus W_e$.

Proof. By permitting: in the construction a number k enters V_e only if number less than or equal to k enters U or enters W_e.

Lemma 3 *For all* e, P_e *is satisfied, that is* $W_e = \Lambda^{V_e}$.

Proof. To determine whether $z \in W_e$ we need to find a stage such that $\lambda(e,z,s)$ has attained its limit. V_e computably determines $\lambda(e,0),\ldots,\lambda(e,z)$ (note that $\lambda(e,y,s)$ changes only if a number $\leq \lambda(e,y,s)$ enters V_e).
Find a stage s_z such that $V_{e,s_z} \upharpoonright \gamma_z + 1 = V_e \upharpoonright \gamma_z + 1$, where $\gamma_z = \max\{\lambda(e,0),\ldots,\lambda(e,z)\}$. Then $z \in W_e$ iff $z \in W_{e,s_z}$.

Lemma 4 V_e *is wtt -mitotic.*
Proof. a) Prove $V_e^0 \leq_{wtt} V_e^1$ (and hence $V_e \leq_{wtt} V_e^1$).
Find a stage s such that $V_e^1 \upharpoonright x-1 = V_{e,s}^1 \upharpoonright x-1$. Then $V_e^0(x) = V_{e,s}^0(x)$.
b) Prove $V_e^0 \leq_T V_e^1$.
Find a stage s such that $V_e^0 \upharpoonright x+1 = V_{e,s}^0 \upharpoonright x+1$. Then $V_e^1(x) = V_{e,s}^1(x)$.

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