On the Structure of Interlaced q-bilattices

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ABSTRACT

Recent developments in logic programming have been based on bilattices. Bilattices were introduced and applied by Ginsberg and Fitting for a diversity of applications, such as truth maintenance systems, default inferences and logic programming. Ginsberg and Fitting proved that every bounded distributive bilattice is the superproduct of two distributive bounded lattices [1]-[5]. A similar result was proved for bounded interlaced bilattices [6] (also see [7]) and then for interlaced bilattices without bounds [8]. In the present paper, we consider more general concepts: the concepts of q-semilattices, q-lattices and q-bilattices, and we find a necessary and sufficient condition for existence of an epimorphism between q-bilattices and the superproduct of two lattices.

Keywords

lattice, bilattice, q-semilattice, q-lattice, q-bilattice.

1. INTRODUCTION

Definition 1. A quasiorder is a reflexive and transitive relation.

Definition 2. The algebra $(L; *)$ with one binary operation is called a q-semilattice if it satisfies the following identities:

$$a * (b * c) = (a * b) * c;$$

$$a * b = b * a;$$

$$a * (b * b) = a * b.$$

The set of all the idempotent elements of the q-semilattice is a semilattice.

Definition 3. The algebra $(L; +, \cdot)$ with two binary operations is called a q-lattice if it satisfies the following identities [9]:

$$a + (b + c) = (a + b) + c; a \cdot (b \cdot c) = (a \cdot b) \cdot c;$$

$$a + b = b + a; a \cdot b = b \cdot a$$

$$a + (b + b) = a + b; a \cdot (b \cdot b) = a \cdot b$$

$$a + (a \cdot a) = a + a; a \cdot (a + b) = a \cdot a;$$

$$a + a = a \cdot a;$$

The set of all the idempotent elements of the q-lattice is a lattice.

Let Q be a quasiorder on $L \neq \emptyset$ set, then $E_Q = Q \bigcap Q^{-1} \subseteq L \times L$ is an equivalent relation. The relation Q/E_Q is an order, which is induced from the Q on L/E_Q set, in the following manner: $(A, B) \in Q/E_Q \Longleftrightarrow aQb$, where $A, B \in L/E_Q$, for any $a \in A$, and $b \in B$. The order Q/E_Q is denoted by $\leq$, and the class which includes the element x is denoted by $[x] \in L/E_Q$.

Definition 4. The function $K : L/E_Q \longrightarrow L$ is called a choice-function if $K([a]) \in [a]$ for every $[a] \in L/E_Q$.

Definition 5. The triple (L, Q, K) is called an inf-quasiordered (sup-quasiordered) set if $\inf([a], [b])$ ($\sup([a], [b])$) exists for each pair of the two elements $[a], [b] \in L/E_Q$.

Definition 6. The triple (L, Q, K) is called an L-quasiordered set if $\inf([a], [b])$ and $\sup([a], [b])$ exist for each pair of the two elements $[a], [b] \in L/E_Q$.

We need the following characterization of q-semilattices.

Theorem 1. Let Q be a quasiorder on the set $L \neq \emptyset$. The following statements are equivalent:

1. $(L/E_Q; \leq)$ is a lower (upper) semilattice;
2. There exists a binary operation $+$ (dual $\cdot$) on the set L such that$(L; +)$ (dual $(L; \cdot)$) is a q-semilattice, which is called a lower (upper) q-semilattice.

   A quasiorder Q is defined by the following rule: $aQb \Longleftrightarrow a + b = a + a (aQb \Longleftrightarrow a \cdot b = b \cdot b)$;

In particular, the following conditions are also equivalent:

1. $(L/E_Q; \leq)$ is a lattice;
2. There exist the binary operations $+$ and $\cdot$ on the set L such that$(L; +, \cdot)$ is a q-lattice.

   And for the quasiorder Q we have: $aQb \Longleftrightarrow a + b = a + a \Longleftrightarrow a \cdot b = b \cdot b$;

Example. Let us consider the oriented graph (L, R), where

$$L = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\},$$

$$R = \{< a_1, a_2 >, < a_2, a_3 >, < a_3, a_1 >, < a_4, a_1 >,$$
$$< a_5, a_3 >, < a_6, a_4 >, < a_7, a_5 >, < a_6, a_7 >, < a_7, a_6 >\}.$$

We say that $aQb \Longleftrightarrow$ if there exists a path from a to b, or $a = b$. The lattice induced from the oriented graph (L, R) is consisted of the following classes:$\{a_1, a_2, a_3\}, \{a_4\}, \{a_5\}, \{a_6, a_7\}$. Let us define the choice-function as the following:

$$K(\{a_1, a_2, a_3\}) = a_2, K(\{a_4\}) = a_4,$$

$$K(\{a_5\}) = a_5, K(\{a_6, a_7\}) = a_6.$$

Hence, we have the L-quasiordered set (L, Q, K) and hence we have the q-lattice $(L; +, \cdot)$ with the operations:

$$x + y = K(\inf([x], [y])), x \cdot y = K(\sup([x], [y])),$$

where $[x], [y] \in L/E_Q$.

So we constructed an example of the q-lattice, which isn't a lattice.

Definition 7. The algebra $(L; +, \cdot, \oplus, \otimes)$ with four binary operations is called q-bilattice (bilattice) if both reducts, $(L; +, \cdot)$ and $(L; \oplus, \otimes)$, are q-lattices (lattices).

The set of all the idempotent elements of the q-bilattice is a bilattice.

Definition 8. Let $L_1 = (L_1; +, \cdot)$ and $L_2 = (L_2; \oplus, \otimes)$ be lattices. The algebra $(L_1 \times L_2; (+, \oplus), (\cdot, \otimes), (+, \otimes), (\cdot, \oplus))$ is called the superproduct of L_1 and L_2 and is denoted by $L_1 \bowtie L_2$.

Definition 9. q-bilattice (bilattice) is interlaced if each of the four basic operations preserves both corresponding quasiorder (order) relations.

2. MAIN RESULT

Theorem 2. For every interlaced q-bilattice $(L; +, \cdot, \oplus, \otimes)$ there exists an epimorphism to the superproduct $L_1 \bowtie L_2$ of two lattices, which is an isomorphism on the bilattice of the idempotent elements.

Finally, we constructed an example of an interlaced q-bilattice for which the corresponding epimorphism is not an isomorphism.

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