Extended symmetric fuzzy constructive logic

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ABSTRACT

A logical system is described based on a symmetry between positive and negative characteristics of situations; such a symmetry is introduced in the concepts of fuzzy logic. Besides, the developed logical theory is treated from the point of view of the constructive (intuitionistic) approach. The notions of strong and weak validity of predicate formulas concerning the introduced logic are defined (i.e. so-called “strong and weak SFCL*-validity” of predicate formulas). The following theorems are formulated: (1) any formula deducible in symmetric constructive predicate calculus HSU* is strongly SFCL*-valid; (2) some formulas having the form (A ⊃ (A ⊃ B)), -(A&¬A), (A&≥ B(x)) ⊃ ≥ A&¬B(x) are not weakly(also not strongly) SFCL*-valid.

Keywords

Fuzzy logic, constructive logic, fuzzy set, recursively enumerable set, REFS-ideal, SREFS-ideal, predicate formula.

In this report the system of Extended symmetric fuzzy constructive logic (shortly, SFCL*) is described. This logic is based on introducing a symmetry between positive and negative logical characteristics of situations in the concepts of fuzzy logic ([19], [29]). Besides, this logic will be considered from the point of view of the constructive (intuitionistic) approach ([6], [12], [13], [14], [16], [17], [20], [22]). The mentioned idea of the symmetry between positive and negative logical characteristics may be clarified as follows. Sometimes (for example, in the investigations of possibilities of artificial intelligence or expert systems) it is necessary to distinguish two kinds of situations: from one side, the situations when we know nothing about the presence of some property p of considered objects, and, from another side, the situations when we know surely that the property p does not take place. For example, establishing medical diagnoses, it is natural to distinguish: (1) the cases when it is quite unknown, whether some illness is present or not in a diagnosis; (2) the cases when it is known surely that the mentioned illness is not present there. Such a logical approach can be formalized, for example, as a logical system using the logical values p satisfying the condition -1 ≤ p ≤ 1. The logical value 1 is interpreted in the framework of the mentioned approach as “the considered property is present”; the logical value 0 is interpreted as “we do not know, whether the considered property is present or not”; the logical value -1 is interpreted as “we know surely that the considered property is not present”. The intermediate logical values describe situations when we know something about the presence or absence of the considered property. An n-dimensional symmetric fuzzy predicate p(x1,x2,...,xn) on a non-empty set M can be defined as a function giving a logical value belonging to [-1,1] for any n-tuple (x1,x2,...,xn), where x1 ∈ M, 1 ≤ i ≤ n. The logical theory of such predicates can be developed, of course, in the framework of the classical set-theoretical approach; however we shall develop similar theory from the points of view of the constructive (intuitionistic) mathematics.

Let us note that there is a standard method for the representation of symmetric fuzzy predicates by fuzzy predicates in the traditional sense of fuzzy logic. Namely, for any symmetric fuzzy predicate p(x1,x2,...,xn) on M we can define its positive component p+ (x1,x2,...,xn) and negative component p- (x1,x2,...,xn) as follows:

p+ (x1,x2,...,xn) = max(0, p(x1,x2,...,xn));
p- (x1,x2,...,xn) = max(0, -p(x1,x2,...,xn)).

Clearly, p and p- are fuzzy predicates in the traditional sense of fuzzy logic. They are disjoint, i.e. p (x1,x2,...,xn) • p(x1,x2,...,xn) = 0 for any n-tuple (x1,x2,...,xn), where x1 ∈ M, 1 ≤ i ≤ n. If we have two disjoint fuzzy predicates q(x1,x2,...,xn) and r(x1,x2,...,xn) on M in the traditional sense of fuzzy logic, then it is easy to see that there exists a symmetric fuzzy predicate p such that

p+ (x1,x2,...,xn) = q(x1,x2,...,xn);
p- (x1,x2,...,xn) = r(x1,x2,...,xn),

for any n-tuple (x1,x2,...,xn), where x1 ∈ M, 1 ≤ i ≤ n. This predicate p is defined uniquely by q and r. So we may consider symmetric fuzzy predicates as pairs of disjoint fuzzy predicates in the sense of fuzzy logic. Similar idea is used when the logical system SFCL* described below is introduced on the base of the “Extended fuzzy constructive logic” (FCL*) described in [5].

Let us recall some definitions. We suppose that the reader is familiar with the theory of recursive functions ([7], [21], [23]) and with the concepts of the classical and constructive (intuitionistic) predicate logic ([18], [21], [28]).

For any n ≥ 1 the n-dimensional recursively enumerable fuzzy set (REFS) is defined as a recursively enumerable set of (n+1)-tuples (x1,x2,...,xn, ε), where all xi are non-negative integers and ε is a binary rational number k 2w, such that 0 ≤ k 2w ≤ 1 (cf. [1], [4], [5], [6],[11], [24],[27]). The n-dimensional REFS w is said to be open if the following conditions hold: (1) if ε = 0 then (x1,x2,...,xn, ε) ∈ w; (2) if (x1,x2,...,xn, ε) ∈ w, and 0 ≤ δ < ε, then (x1,x2,...,xn, δ) ∈ w; (3) for any (n+1)-tuple (x1,x2,...,xn, ε) ∈ w, where ε > 0, there exists such δ > ε, that (x1,x2,...,xn, δ) ∈ w (cf. [4], [5]). We shall consider below, as a rule, only open REFSes; some exceptions will be noted apart.

The notion of pseudonumber is defined as in [6]. The Gödel numbering of pseudonumbers is defined similarly to the Gödel numbering of the constructive real numbers ([6], [17]). Specker’s number ([6]) is a pseudonumber defined by a non-decreasing constructive sequence of binary rational numbers. The Specker’s representation of an n-dimensional REFS w (not obligatory open) is defined as a general recursive function of n variables satisfying the following conditions: if for some n-tuple (x1,x2,...,xn) of non-
negative integers there exists no such ε that (x₁, x₂,..., xᵣ, ε) ∈ w then this general recursive function gives for this (x₁, x₂,..., xᵣ) a Gödel number of a Specker’s number which is equal to 0; in the opposite case it gives a Gödel number of a Specker’s number Ψᵥ(x₁,x₂,...,xᵣ) which is the supremum of binary rational ε such that (x₁, x₂,..., xᵣ, ε) ∈ w. The Specker’s standard function is defined as a general recursive function satisfying the following conditions: for any n-tuple (x₁,x₂,..., xᵣ) of non-negative integers it gives a Gödel number of a Specker’s number Ψᵥ(x₁,x₂,...,xᵣ) such that 0 ≤ Ψᵥ(x₁,x₂,...,xᵣ) ≤ 1. It is easy to see that for any Specker’s standard function Ψᵥ(x₁,x₂,...,xᵣ) there exists an n-dimensional open REFS w such that
Ψᵥ(x₁,x₂,...,xᵣ) = Ψᵥ(x₁,x₂,...,xᵣ)
for any n-tuple (x₁,x₂,...,xᵣ) of non-negative integers. It is easy to verify that there is a constructive one-to-one correspondence between the Specker’s standard n-dimensional functions and n-dimensional open REFSes. We say that an n-dimensional REFS w covers an n-dimensional REFS u and write u ⊆ w if for any
Ψᵥ(x₁,x₂,...,xᵣ) ≤ Ψᵥ(x₁,x₂,...,xᵣ).
We say that n-dimensional REFSes w and u are equivalent and write w = u, if w covers u, and u covers w. This notion of equivalence is used in [4], [5], [24]; it is different from the notion of equivalence used in [10], [11], [25]-[27]. It is easy to check that the mentioned two notions of equivalence coincide for open REFSes. It is easy to see also that the relations “w covers u”, “w is equivalent to u” coincide for open REFSes with the relations ⊆ and ≡ interpreted from the usual set-theoretical points of view. The operations of union ∪ and intersection ∩ of n-dimensional REFSes are defined in a usual way; it is easy to see, that
Ψᵥ(∪,x₁,x₂,...,xᵣ) = min(Ψᵥ(x₁,x₂,...,xᵣ), Ψᵥ(x₁,x₂,...,xᵣ), Ψᵥ(x₁,x₂,...,xᵣ))
Ψᵥ(∪,x₁,x₂,...,xᵣ) = max(Ψᵥ(x₁,x₂,...,xᵣ), Ψᵥ(x₁,x₂,...,xᵣ), Ψᵥ(x₁,x₂,...,xᵣ))
for any open n-dimensional REFSes w and u, and for any x₁,x₂,...,xᵣ.
The operation of Cartesian product w × u of REFSes w and u, the operation of projection ↓ᵢ (w) of an n-dimensional REFS w on i-th coordinate (where 1 ≤ i ≤ n), the operation of transposition Tᵢ (w) of i-th and j-th coordinates in an n-dimensional REFS w (where 1 ≤ i,j ≤ n) are defined as in [27] (cf. also [4], [5], [10], [11], [26]). The operation of generalization ↑ᵢ (w) of an n-dimensional REFS w on i-th coordinate (where 1 ≤ i ≤ n) is defined as the operation of constructing an n-dimensional open REFS y such that for any n-tuple (x₁, x₂,..., xᵣ) of non-negative integers
Ψᵥ(x₁,x₂,...,xᵣ) = Ψᵥ(x₁,x₂,...,xᵣ)
where u = ↓ᵢ (w) (cf. [4], [5]). The operation of substitution Subᵢ (w) of the variable xᵢ for the variable xᵢ in an n-dimensional REFS w (where 1 ≤ i,j ≤ n) is defined as the operation of constructing an open n-dimensional REFS y such that for any n-tuple (x₁,x₂,...,xᵣ) of non-negative integers
Ψᵥ(x₁,x₂,...,xᵣ) = Ψᵥ(x₁,x₂,...,xᵣ)
(cf. [4], [5]).

An n-dimensional REFS Vⁿ (correspondingly, Δⁿ) is defined as an open n-dimensional REFS, containing all the (n+1)-tuples (x₁,x₂,..., xᵣ, ε), such that 0 ≤ ε<1 (correspondingly, containing only such (n+1)-tuples (x₁,x₂,..., xᵣ, ε), where ε = 0).

The n-dimensional REFS-ideal is defined as a non-empty set Δ of n-dimensional open REFSes such that the following conditions hold:
1) if w ∈ Δ, and u ⊆ w, then u ∈ Δ;
2) if w ∈ Δ, and u ∈ Δ, then w ∪ u ∈ Δ (cf. [5]).

An n-dimensional REFS-ideal is said to be principal ideal, if there exists an n-dimensional open REFS w₀ such that w ∈ Δ if and only if w ⊆ w₀ (cf. [5]).

An n-dimensional REFS-ideal Δ is said to be complete if all n-dimensional open REFSes belong to Δ. Clearly, Δ is complete if and only if Vⁿ ⊆ Δ (cf. [5]).

An n-dimensional REFS-ideal is said to be null-ideal (or null-REFS-ideal) if w ∈ Δ only for w = Aⁿ (cf. [5]).

Let Δ be a non-empty set of n-dimensional open REFSes. The n-dimensional REFS-ideal Δ generated by the set Δ is defined as the set Δ satisfying the following conditions: w ∈ Δ if and only if there exists k-tuple of n-dimensional open REFSes (u₁,u₂,...,uₖ) such that uᵢ ∈ Δ, 1 ≤ i ≤ k, and w ⊆ u₁ ∪ u₂ ∪ ... ∪ uₖ. It is easy to see that the set Δ defined by such a way is a REFS-ideal (cf. [5]).

Two n-dimensional REFS-ideals Δ₁ and Δ₂ are said to be disjoint if w₁ ∩ w₂ = Δ for any w₁ ∈ Δ₁ and w₂ ∈ Δ₂.

The n-dimensional SREFS-ideal Δ ⊆ Δⁿ is defined as any pair of disjoint n-dimensional REFS-ideals Δ₁ and Δ₂. The components Δ₁ and Δ₂ are said to be positive component Δ₁ and negative component Δ₂ of the SREFS-ideal Δ = (Δ₁, Δ₂).

An n-dimensional SREFS-ideal Δ = (Δ₁, Δ₂) is said to be principal if Δ₁ and Δ₂ are principal REFS-ideals.

An n-dimensional SREFS-ideal Δ = (Δ₁, Δ₂) is said to be complete if Δ₁ is a complete REFS-ideal, and Δ₂ is a null-REFS-ideal.

Let Δ₁ and Δ₂ be non-empty sets of n-dimensional open REFSes such that w ∩ u = Δ for any w ∈ Δ₁ and u ∈ Δ₂. The n-dimensional SREFS-ideal Δ = (Δ₁, Δ₂) generated by the pair of sets (Δ₁, Δ₂) is defined as the pair of sets (Δ₁, Δ₂) satisfying the following conditions: Δ₁ is the REFS-ideal generated by Δ₁, and Δ₂ is the REFS-ideal generated by Δ₂. It is easy to see that the pair of sets (Δ₁, Δ₂) defined by such a way is a SREFS-ideal.

We consider the language of predicate formulas which are constructed by the logical operations & , ∨ , ¬ , → , ∃ , ∀ , Ǝ , and do not contain functional symbols and symbols of constants. We suppose that this language contains an infinite (enumerable) set of n-dimensional predicate symbols for any n ≥ 1. The symbol T of truth, the symbol F of falsity, and the symbol U of uncertainty are included in the set of elementary formulas. All the definitions connected with the predicate formulas are given in the natural way ([18], [21]).

We suppose (as in [4], [5]) that a sequence x₁,x₂,... containing all variables of the considered language is fixed. For any formula A its index majorant is
defined as any positive integer $k$ such that $k \geq m$ for any index $m$ of a variable $x_m$ (free or not free) occurring in $A$.

Let $A$ be a predicate formula which contains only predicate symbols $p_1, p_2, \ldots, p_n$ having the dimensions, corresponding to $i_1, i_2, \ldots, i_n$.

A SFCL*-assignment for $A$ is defined as a correspondence assigning to any $p_i$ where $1 \leq k \leq l$, some $i_k$-dimensional SREFS-ideal.

A SFCL*-assignment is said to be principal if all the SREFS-ideals assigned to $p_1, p_2, \ldots, p_n$ are principal.

We define SFCL*-interpretation $\Pi_{\phi,k}(A)$ of a given formula $A$ concerning a SFCL*-assignment $\phi$ for $A$ and an index majorant $k$ of $A$. For any $\phi, k$, the SFCL*-interpretation $\Pi_{\phi,k}(A)$ is defined as some k-dimensional SREFS-ideal; its positive and negative component will be denoted as, correspondingly, $\Pi_+^{\phi,k}(A)$ and $\Pi_-^{\phi,k}(A)$. The definition of $\Pi_{\phi,k}(A)$ is given by induction on the construction of $A$. Let $A$ be an elementary formula having the form $p(x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n$ are variables $x_i$ with the indices $i_1, i_2, \ldots, i_n$. Let $\Delta$ be a $k$-dimensional SFRES-ideal assigned to $p_i$ in $\phi$. The SFCL*-interpretation $\Pi_{\phi,k}(A)$ is constructed as follows. By $\Delta'$ and $\Delta''$ we denote $k$-dimensional FRES-ideals generated by all SREFSes having the form $\text{w} \times A'$, where, correspondingly, $\text{w} \in \Delta'$ or $\text{w} \in \Delta''$. Clearly, $\Delta'$ and $\Delta''$ are disjoint, so they can be considered as components of an SREFS-ideal $\Delta$. On the base of $\Delta'$ and $\Delta''$ we construct now SREFS-ideals $\Delta'$ and $\Delta''$ generated by all SREFSes, correspondingly, $\text{w}_1$ and $\text{w}_2$ which are obtained from SREFSes $\gamma_1 \in \Delta'$ and $\gamma_2 \in \Delta''$ by a sequence of operations $\Gamma_{\phi,k}$ and $\Gamma_{\phi,k}$ (the same for $\gamma_1 \in \Delta'$ and $\gamma_2 \in \Delta''$) displacing the variables $x_1, x_2, \ldots, x_n$ to the positions corresponding to the indices $i_1, i_2, \ldots, i_n$ (the existence of such sequence of operations is easy to see).

Clearly, $\Delta'$ and $\Delta''$ are disjoint; we define $\Pi_+^{\phi,k}(A)$ and $\Pi_-^{\phi,k}(A)$ as, correspondingly, $\Delta'$ and $\Delta''$. For the elementary formulas $T, F$, and $U$ we define $\Pi_+^{\phi,k}(T)$ and $\Pi_-^{\phi,k}(U)$ as complete k-dimensional SFRES-ideals;

$\Pi_+^{\phi,k}(T)$, $\Pi_+^{\phi,k}(F)$, $\Pi_-^{\phi,k}(U)$ and $\Pi_-^{\phi,k}(U)$ are defined as k-dimensional null-SREFS-ideals. For non-elementary formulas the SREFS-ideals of $\Pi_{\phi,k}(A)$ are defined in the following way.

1. $\Pi_+^{\phi,k}(A \& B)$ is the set of all open SREFSes having the form $w \cup u$, where $w \in \Pi_+^{\phi,k}(A)$, $u \in \Pi_+^{\phi,k}(B)$; $\Pi_-^{\phi,k}(A \& B)$ is the set of all open SREFSes having the form $w \cup u$, where $w \in \Pi_-^{\phi,k}(A)$, $u \in \Pi_-^{\phi,k}(B)$.

2. $\Pi^+_\varphi(A \lor B)$ is the set of all open SREFSes having the form $w \cup u$, where $w \in \Pi^+_\varphi(A)$, $u \in \Pi^+_\varphi(B)$; $\Pi^-_\varphi(A \lor B)$ is the set of all open SREFSes having the form $w \cup u$, where $w \in \Pi^-_\varphi(A)$, $u \in \Pi^-_\varphi(B)$.

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