Extended symmetric fuzzy constructive logic

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ABSTRACT

A logical system is described based on a symmetry between positive and negative characteristics of situations; such a symmetry is introduced in the concepts of fuzzy logic. Besides, the developed logical theory is treated from the point of view of the constructive (intuitionistic) approach. The notions of strong and weak validity of predicate formulas concerning the introduced logic are defined (i.e. so-called "strong and weak SFCL*-validity" of predicate formulas). The following theorems are formulated: (1) any formula deducible in symmetric constructive predicate calculus HSU′ is strongly SFCL*-valid; (2) some formulas having the form $(A \supset (A \supset B)) \supset (A \supset B)$, $\neg(A\&\neg A)$, $(A\&\exists x\, B(x)) \supset \exists x(A\&B(x))$ are not weakly(also not strongly) SFCL*-valid.

Keywords

Fuzzy logic, constructive logic, fuzzy set, recursively enumerable set, REFS-ideal, SREFS-ideal, predicate formula.

In this report the system of Extended symmetric fuzzy constructive logic (shortly, SFCL*) is described. This logic is based on introducing a symmetry between positive and negative logical characteristics of situations in the concepts of fuzzy logic ([19], [29]). Besides, this logic will be considered from the point of view of the constructive (intuitionistic) approach ([6], [12], [13], [14], [16], [17], [20], [22]). The mentioned idea of the symmetry between positive and negative logical characteristics may be clarified as follows. Sometimes(for example, in the investigations of possibilities of artificial intelligence or expert systems) it is necessary to distinguish two kinds of situations: from one side, the situations when we know nothing about the presence of some property p of considered objects, and, from another side, the situations when we know surely that the property p does not take place. For example, establishing medical diagnoses, it is natural to distinguish: (1) the cases when it is quite unknown, whether some illness is present or not in a diagnosis; (2) the cases when it is known surely that the mentioned illness is not present there. Such a logical approach can be formalized, for example, as a logical system using the logical values p satisfying the condition $-1 \leq p \leq 1$. The logical value 1 is interpreted in the framework of the mentioned approach as "the considered property is present"; the logical value 0 is interpreted as "we do not know, whether the considered property is present or not"; the logical value -1 is interpreted as "we know surely that the considered property is not present". The intermediate logical values describe situations when we know something about the presence or absence of the considered property. An n-dimensional symmetric fuzzy predicate $p(x_1,x_2,...,x_n)$ on a non-empty set M can be defined as a function giving a logical value belonging to [-1,1] for any n-tuple $(x_1,x_2,...,x_n)$, where $x_i \in M$, $1 \leq i \leq n$. The logical theory of such predicates can be developed, of course, in the framework of the classical set-theoretical approach; however we shall develop similar theory from the points of view of the constructive (intuitionistic) mathematics.

Let us note that there is a standard method for the representation of symmetric fuzzy predicates by fuzzy predicates in the traditional sense of fuzzy logic. Namely, for any symmetric fuzzy predicate $p(x_1, x_2,...,x_n)$ on M we can define its positive component $p^+(x_1,x_2,...,x_n)$ and negative component $p^-(x_1, x_2,...,x_n)$ as follows:

$$p^+(x_1, x_2,...,x_n) = \max(0, p(x_1, x_2,...,x_n));$$
$$p^-(x_1, x_2,...,x_n) = \max(0, -p(x_1, x_2,...,x_n)).$$

Clearly, p^+ and p^- are fuzzy predicates in the traditional sense of fuzzy logic. They are disjoint, i.e. $p^+(x_1,x_2,...,x_n) \bullet p^-(x_1,x_2,...,x_n) = 0$ for any n-tuple $(x_1, x_2,...,x_n)$, where $x_i \in M$, $1 \leq i \leq n$. If we have two disjoint fuzzy predicates $q(x_1, x_2,...,x_n)$ and $r(x_1, x_2,...,x_n)$ on M in the traditional sense of fuzzy logic, then it is easy to see that there exists a symmetric fuzzy predicate p such that

$$p^+(x_1,x_2,...,x_n) = q(x_1,x_2,...,x_n);$$
$$p^-(x_1, x_2,...,x_n) = r(x_1, x_2,...,x_n),$$

for any n-tuple $(x_1,x_2,...,x_n)$, where $x_i \in M$, $1 \leq i \leq n$. This predicate p is defined uniquely by q and r. So we may consider symmetric fuzzy predicates as pairs of disjoint fuzzy predicates in the sense of fuzzy logic. Similar idea is used when the logical system SFCL* described below is introduced on the base of the "Extended fuzzy constructive logic" (FCL*) described in [5].

Let us recall some definitions. We suppose that the reader is familiar with the theory of recursive functions ([7], [21], [23]) and with the concepts of the classical and constructive (intuistionistic) predicate logic ([18], [21], [28]).

For any $n \geq 1$ the n-dimensional recursively enumerable fuzzy set (REFS) is defined as a recursively enumerable set of (n+1)-tuples $(x_1, x_2,...,x_n, \varepsilon)$, where all x_i are non-negative integers and ε is a binary rational number $\frac{k}{2^m}$, such that $0 \leq \frac{k}{2^m} \leq 1$ (cf. [1], [4], [5], [8]-[11], [24]-[27]). The n-dimensional REFS w is said to be open if the following conditions hold: (1) if ε =0 then $(x_1,x_2,...,x_n, \varepsilon) \in w$; (2) if $(x_1,x_2,...,x_n, \varepsilon) \in w$, and $0 \leq \delta < \varepsilon$, then $(x_1,x_2,...,x_n, \delta) \in w$; (3) for any (n+1)-tuple $(x_1,x_2,...,x_n, \varepsilon) \in w$, where $\varepsilon > 0$,there exists such $\delta > \varepsilon$, that $(x_1,x_2,...,x_n, \delta) \in w$ (cf. [4], [5]). We shall consider below, as a rule, only open RFESes; some exceptions will be noted apart.

The notion of pseudonumber is defined as in [6]. The Gödel numbering of pseudonumbers is defined similarly to the Gödel numbering of the constructive real numbers ([6], [17]). Specker's number ([6]) is a pseudonumber defined by a non-decreasing constructive sequence of binary rational numbers. The Specker's representation of an n-dimensional REFS w (not obligatory open) is defined as a general recursive function of n variables satisfying the following conditions: if for some n-tuple $(x_1,x_2,...,x_n)$ of non-negative integers there exists no such ε that $(x_1,x_2,...,x_n, \varepsilon) \in w$ then this general recursive function gives for this $(x_1,x_2,...,x_n)$ a Gödel number of a Specker's number which is equal to 0; in the opposite case it gives a Gödel number of a Specker's number $\Psi_w(x_1,x_2,...,x_n)$ which is the supremum of binary rational ε such that $(x_1,x_2,...,x_n, \varepsilon) \in w$. The Specker's standard function is defined as a general recursive function satisfying the following conditions: for any n-tuple $(x_1,x_2,...,x_n)$ of non-negative integers it gives a Gödel number of a Specker's number $\Psi(x_1,x_2,...,x_n)$ such that $0 \leq \Psi(x_1,x_2,...,x_n) \leq 1$. It is easy to see that for any Specker's standard function $\Psi(x_1,x_2,...,x_n)$ there exists an n-dimensional open REFS w such that

$$\Psi_w(x_1,x_2,...,x_n) = \Psi(x_1,x_2,...,x_n)$$

for any n-tuple $(x_1,x_2,...,x_n)$ of non-negative integers. It is easy to verify that there is a constructive one-to one correspondence between the Specker's standard n-dimensional functions and n-dimensional open REFSes.

We say that an n-dimensional REFS w covers an n-dimensional REFS u and write $u \subseteq w$ if for any $x_1,x_2,...,x_n$

$$\Psi_u(x_1,x_2,...,x_n) \leq \Psi_w(x_1,x_2,...,x_n).$$

We say that n-dimensional REFSes w and u are equivalent and write w=u, if w covers u, and u covers w. This notion of equivalence is used in [4], [5], [24]; it is different from the notion of equivalence used in [10], [11], [25]-[27]. It is easy to check that the mentioned two notions of equivalence coincide for open REFSes. It is easy to see also that the relations " w covers u", "w is equivalent to u" coincide for open REFSes with the relations $\subseteq$ and = interpreted from the usual set-theoretical points of view.

The operations of union $\cup$ and intersection $\cap$ of n-dimensional REFSes are defined in an usual way; it is easy to see, that

$$\Psi_{w \cap u}(x_1,x_2,...,x_n) = \min(\Psi_w(x_1,x_2,...,x_n), \Psi_u(x_1,x_2,...,x_n)),$$

$$\Psi_{w \cup u}(x_1,x_2,...,x_n) = \max(\Psi_w(x_1,x_2,...,x_n), \Psi_u(x_1,x_2,...,x_n))$$

for any open n-dimensional REFSes w and u, and for any $x_1,x_2,...,x_n$.

The operation of Cartesian product $w \times u$ of REFSes w and u, the operation or projection $\downarrow_i^n(w)$ of an n-dimensional REFS w on i-th coordinate (where $1 \leq i \leq n$), the operation of transposition $T_{ij}^n(w)$ of i-th and j-th coordinates in an n-dimensional REFS w (where $1 \leq i,j \leq n$) are defined as in [27] (cf. also [4], [5], [10], [11], [26]). The operation of generalization $\uparrow_i^n(w)$ of an n-dimensional REFS w on i-th coordinate (where $1 \leq i \leq n$) is defined as the operation of constructing an n-dimensional open REFS y such that for any n-tuple $(x_1, x_2,...,x_n)$ of non-negative integers

$$\Psi_y(x_1,x_2,...,x_{i-1},x_i,x_{i+1},...,x_n) = \Psi_u(x_1,x_2,...,x_{i-1},x_{i+1},...,x_n),$$

where $u = \downarrow_i^n(w)$ (cf. [4], [5]). The operation of substitution $\mathrm{Sub}_{ij}^n(w)$ of the variable x_j for the variable x_i in an n-dimensional REFS w (where $1 \leq i,j \leq n$) is defined as the operation of constructing an n-dimensional open REFS y such that for any n-tuple $(x_1,x_2,...,x_n)$ of non-negative integers

$$\Psi_y(x_1,x_2,...,x_{i-1},x_i,x_{i+1},...,x_{j-1},x_j,x_{j+1},...,x_n) = \Psi_w(x_1,x_2,...,x_{i-1},x_j,x_{i+1},...,x_{j-1},x_j,x_{j+1},...,x_n),$$

(cf. [4], [5]).

An n-dimensional REFS V^n (correspondingly, Λ^n) is defined as an open n-dimensional REFS, containing all the (n+1)-tuples $(x_1,x_2,...,x_n, \varepsilon)$, such that $0 \leq \varepsilon < 1$ (correspondingly, containing only such (n+1)-tuples $(x_1,x_2,...,x_n, \varepsilon)$, where $\varepsilon = 0$).

The n-dimensional REFS-ideal is defined as a non-empty set Δ of n-dimensional open REFSes such that the following conditions hold:

1) if $w \in \Delta$, and $u \subseteq w$, then $u \in \Delta$;
2) if $w \in \Delta$, and $u \in \Delta$, then $w \cup u \in \Delta$

(cf. [5]).

An n-dimensional REFS-ideal is said to be principal ideal, if there exists an n-dimensional open REFS w_0 such that $w \in \Delta$ if and only if $w \subseteq w_0$ (cf. [5]).

An n-dimensional REFS-ideal Δ is said to be complete if all n-dimensional open REFSes belong to Δ. Clearly, Δ is complete if and only if $V^n \in \Delta$ (cf. [5]).

An n-dimensional REFS-ideal Δ is said to be null-ideal (or null-REFS-ideal) if $w \in \Delta$ only for $w = \Lambda^n$ (cf. [5]).

Let Δ be a non-empty set of n-dimensional open REFSes. The n-dimensional REFS-ideal Δ' generated by the set Δ is defined as the set Δ' satisfying the following condition: $w \in \Delta'$ if and only if there exists a k-tuple of n-dimensional open REFSes $(u_1,u_2,...,u_k)$ such that $u_i \in \Delta$, $1 \leq i \leq k$, and $w \subseteq u_1 \cup u_2 \cup ... \cup u_k$. It is easy to see that the set Δ' defined by such a way is a REFS-ideal (cf. [5]).

Two n-dimensional REFS-ideals Δ_1 and Δ_2 are said to be disjoint if $w_1 \cap w_2 = \Lambda^n$ for any $w_1 \in \Delta_1$ and $w_2 \in \Delta_2$.

The n-dimensional SREFS-ideal $\Delta=(\Delta^+,\Delta^-)$ is defined as any pair of disjoint n-dimensional REFS-ideals Δ^+ and Δ^-. The components Δ^+ and Δ^- are said to be positive component Δ^+ and negative component Δ^- of the SREFS-ideal $\Delta = (\Delta^+,\Delta^-)$.

An n-dimensional SREFS-ideal $\Delta=(\Delta^+,\Delta^-)$ is said to be principal if Δ^+ and Δ^- are principal REFS-ideals.

An n-dimensional SREFS-ideal $\Delta=(\Delta^+,\Delta^-)$ is said to be complete if Δ^+ is a complete REFS-ideal, and Δ^- is a null-REFS-ideal.

Let Δ_1 and Δ_2 be non-empty sets of n-dimensional open REFSes such that $w \cap u = \Lambda^n$ for any $w \in \Delta_1$ and $u \in \Delta_2$. The n-dimensional SREFS-ideal $\Delta=(\Delta^+,\Delta^-)$ generated by the pair of sets (Δ_1,Δ_2) is defined as the pair of sets (Δ^+,Δ^-) satisfying the following conditions: Δ^+ is the REFS-ideal generated by Δ_1, and Δ^- is the REFS-ideal generated by Δ_2. It is easy to see that the pair of sets (Δ^+,Δ^-) defined by such a way is a SREFS-ideal.

We consider the language of predicate formulas which are constructed by the logical operations &, $\vee$, $\supset$, $\neg$, $\forall$, $\exists$, and do not contain functional symbols and symbols of constants. We suppose that this language contains an infinite (enumerable) set of n-dimensional predicate symbols for any $n \geq 1$. The symbol T of truth, the symbol F of falsity, and the symbol U of uncertainty are included in the set of elementary formulas. All the definitions connected with the predicate formulas are given in the natural way ([18], [21]).

We suppose (as in [4], [5]) that a sequence $x_1,x_2,...$ containing all variables of the considered language is fixed. For any formula A its index majorant is defined as any positive integer k such that k $\geq$ m for any index m of a variable x_m (free or not free) occurring in A.

Let A be a predicate formula which contains only predicate symbols $p_1, p_2, ..., p_l$ having the dimensions, correspondingly, $i_1, i_2, ..., i_l$.

A SFCL*-assignment for A is defined as a correspondence assigning to any p_k, where $1 \leq k \leq l$, some i_k-dimensional SREFS-ideal.

A SFCL*-assignment is said to be principal if all the SREFS-ideals assigned to $p_1, p_2, ..., p_l$ are principal.

Let us define SFCL*-interpretation $\Pi_{\varphi,k}(A)$ of a given formula A concerning a SFCL*-assignment φ for A and an index majorant k of A. For any A, φ, k the SFCL*-interpretation $\Pi_{\varphi,k}(A)$ is defined as some k-dimensional SREFS-ideal; its positive and negative component will be denoted as, correspondingly, $\Pi^+_{\varphi,k}(A)$ and $\Pi^-_{\varphi,k}(A)$. The definition of $\Pi_{\varphi,k}(A)$ is given by induction on the construction of A. Let A be an elementary formula having the form $p_r(\xi_1, \xi_2, ..., \xi_t)$, where $\xi_1, \xi_2, ..., \xi_t$ are variables x_j with the indices $j_1, j_2, ..., j_t$. Let Δ be a t-dimensional SFRES-ideal assigned to p_r in φ. The SFCL*-interpretation $\Pi_{\varphi,k}(A)$ is constructed as follows. By Δ'^+ and Δ'^- we denote k-dimensional FRES-ideals generated by all REFSes having the form $w \times V^{k-t}$, where, correspondingly, $w \in \Delta^+$ or $w \in \Delta^-$. Clearly, Δ'^+ and Δ'^- are disjoint, so they can be considered as components of an SREFS-ideal Δ'. On the base of Δ'^+ and Δ'^- we construct now REFS-ideals Δ''^+ and Δ''^- generated by all REFSes, correspondingly, w_1 and w_2 which are obtained from REFSes $y_1 \in \Delta'^+$ and $y_2 \in \Delta'^-$ by a sequence of operations T^k_{ij} and Sub^k_{ij} (the same for all $y_1 \in \Delta'^+$ and $y_2 \in \Delta'^-$) displacing the variables $x_1, x_2, ..., x_t$ to the positions corresponding to the indices $j_1, j_2, ..., j_t$ (the existence of such sequence of operations is easy to see). Clearly, Δ''^+ and Δ''^- are disjoint; we define $\Pi^+_{\varphi,k}(A)$ and $\Pi^-_{\varphi,k}(A)$ as, correspondingly, Δ''^+ and Δ''^-. For the elementary formulas T, F, and U we define $\Pi^+_{\varphi,k}(T)$ and $\Pi^-_{\varphi,k}(F)$ as complete k-dimensional REFS-ideals; $\Pi^-_{\varphi,k}(T)$, $\Pi^+_{\varphi,k}(F)$, $\Pi^+_{\varphi,k}(U)$ and $\Pi^-_{\varphi,k}(U)$ are defined as k-dimensional null-REFS-ideals. For non-elementary formulas the SREFS-ideals $\Pi_{\varphi,k}(A)$ are defined in the following way.

(1) $\Pi^+_{\varphi,k}(A\&B)$ is the set of all open REFSes having the form $w \cap u$, where $w \in \Pi^+_{\varphi,k}(A)$, $u \in \Pi^+_{\varphi,k}(B)$; $\Pi^-_{\varphi,k}(A\&B)$ is the set of all open REFSes having the form $w \cup u$, where $w \in \Pi^-_{\varphi,k}(A)$, $u \in \Pi^-_{\varphi,k}(B)$.

(2) $\Pi^+_{\varphi,k}(A \vee B)$ is the set of all open REFSes having the form $w \cup u$, where $w \in \Pi^+_{\varphi,k}(A)$, $u \in \Pi^+_{\varphi,k}(B)$; $\Pi^-_{\varphi,k}(A \vee B)$ is the set of all open REFSes having the form $w \cap u$, where $w \in \Pi^-_{\varphi,k}(A)$, $u \in \Pi^-_{\varphi,k}(B)$.

(3) $\Pi^+_{\varphi,k}(\neg A)$ is $\Pi^-_{\varphi,k}(A)$; $\Pi^-_{\varphi,k}(\neg A)$ is $\Pi^+_{\varphi,k}(A)$;

(4) $\Pi^+_{\varphi,k}(A \supset B)$ is the set of all k-dimensional open REFSes w satisfying the following conditions: $w \cap u \in \Pi^+_{\varphi,k}(B)$ for any $u \in \Pi^+_{\varphi,k}(A)$, and $w \cap u \in \Pi^-_{\varphi,k}(A)$ for any $u \in \Pi^-_{\varphi,k}(B)$.

(5) $\Pi^-_{\varphi,k}(A \supset B)$ is the set of all open REFSes having the form $w \cap u$, where $w \in \Pi^+_{\varphi,k}(A)$, $u \in \Pi^-_{\varphi,k}(B)$.

(6) $\Pi^+_{\varphi,k}(\exists x_i(A))$ is the set of all k-dimensional open REFSes w satisfying the following condition: there exists an open k-dimensional REFS $u \in \Pi^+_{\varphi,k}(A)$ such that $w \subseteq \uparrow^n_i(u)$; $\Pi^-_{\varphi,k}(\exists x_i(A))$ is the set of all k-dimensional open REFSes w such that $\uparrow^k_i(w) \subseteq \Pi^-_{\varphi,k}(A)$.

(7) $\Pi^+_{\varphi,k}(\forall x_i(A))$ is the set of all k-dimensional open REFSes w such that $\uparrow^k_i(w) \subseteq \Pi^+_{\varphi,k}(A)$; $\Pi^-_{\varphi,k}(\forall x_i(A))$ is the set of all k-dimensional open REFSes w satisfying the following condition: there exists an open k-dimensional REFS $u \in \Pi^-_{\varphi,k}(A)$ such that $w \subseteq \uparrow^n_i(u)$.

It is easy to verify that in all cases the REFS-ideals $\Pi^+_{\varphi,k}(A)$ and $\Pi^-_{\varphi,k}(A)$ described in the definitions given above, are disjoint REFS-ideals. So they form a SREFS-ideal $\Pi_{\varphi,k}(A)$.

We say that a predicate formula A is strongly SFCL*-valid (correspondingly, weakly SFCL*-valid) if there exists such k_0, that for every SFCL*-assignment φ (correspondingly, for every principal SFCL*-assignment φ) and for $k \geq k_0$ the interpretation $\Pi_{\varphi,k}(A)$ is complete. This notion actually defines the considered semantic of SFCL*.

The symmetric constructive predicate calculus HSU is defined as in [3]. The calculus HSU' is obtained from HSU by excluding the functional symbols and the symbols of constants from the language of HSU (the logical constants T, F, U remain in the language). The form of axioms and of rules of inference for HSU' is the same as for HSU.

Theorem 1. Every predicate formula deducible in HSU' is strongly SFCL*-valid.

Theorem 2. The formulas $(p(x_1) \supset (p(x_1) \supset q(x_1))) \supset (p(x_1) \supset q(x_1))$, $\neg(p(x_1)\&\neg p(x_1))$, $(p(x_1)\&\exists x_2(q(x_1,x_2))) \supset \exists x_2(p(x_1)\&q(x_1,x_2))$ are not weakly SFCL*-valid.

Let us note that the formulas mentioned in the formulation of Theorem 2 are not deducible in HSU'.

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