Comparison of the Efficiency of Frege Systems with Restricted Substitution Rules

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ABSTRACT

We compare the proof complexities in Frege systems with substitution rule without any restrictions and with depth-restricted substitution rule. We prove that any two depth-restricted substitution Frege systems are polynomially equivalent both by size and by steps. Frege system with well-known substitution rule and Frege system with depth-restricted substitution rule are also polynomially equivalent by size, but the first system has exponential speed-up over the second system by steps.

Keywords

Frege system, proof complexity, depth-restricted substitution rule, polynomially equivalent, exponential speed-up.

1. INTRODUCTION

One of the most fundamental problems of the complexity theory is to find an efficient proof system for propositional calculus. First, we have to make it clear what the notion "efficient" means. There is a wide spread understanding that polynomial time computability is the correct mathematical model of feasible computation. According to the opinion, a truly "effective" system must have a polynomial size, $p(n)$ proof for every tautology of size n. In [1] Cook and Reckhow named such a system, a *super* system. They showed that if there exists a super system, then $NP = coNP$.

It is well known that many systems are not super. This question about Frege systems, the most natural calculi for propositional logic, is still open. It is interesting how efficient can be Frege systems augmented with new, not sound rules, in particular – Frege systems with different modifications of substitution rules.

It is known that a Frege system with substitution rule has exponential speed-up by steps over the Frege system without substitution rule [2]. It is known also that Frege system with multiple substitution rule has exponential speed-up by steps over the Frege system with single substitution rule [3]. In this paper a depth-restricted substitution rule is introduced and any two depth-restricted substitution Frege systems as well as the Frege systems with substitution rule without restrictions and with depth-restricted substitution rule are compared.

We prove that

1) the minimal numbers of steps (the minimal sizes) of the proofs of tautology φ in any two depth-restricted substitution Frege systems are polynomially related;
2) the minimal sizes of the proofs of tautology φ in a without restrictions substitution Frege system and in a depth-restricted substitution Frege system are also polynomially related;
3) the minimal number of steps of a tautology in a depth-restricted substitution Frege system can be exponentially larger than in the system with substitution rule without restrictions.

2. MAIN NOTIONS AND NOTATIONS

We shall use generally accepted concepts of Frege system and Frege system with substitution.

A Frege system $\mathcal{F}$ uses a denumerable set of propositional variables, a finite, complete set of propositional connectives; $\mathcal{F}$ has a finite set of inference rules defined by a *figure* of the form $\frac{A_1 A_2 \ldots A_k}{B}$ (the rules of inference with zero hypotheses are the axioms schemes); $\mathcal{F}$ must be sound and complete, i.e. for each rule of inference $\frac{A_1 A_2 \ldots A_k}{B}$ every truth-value assignment satisfying $A_1, A_2, \ldots, A_k$ also satisfies B, and $\mathcal{F}$ must prove every tautology.

A *substitution Frege system* $S\mathcal{F}$ consists of a Frege system $\mathcal{F}$ augmented with the substitution rule with inferences of the form $\frac{A}{A\sigma}$ for any substitution $\sigma = \begin{pmatrix} \varphi_{i_1} & \varphi_{i_2} & \ldots & \varphi_{i_s} \\ p_{i_1} & p_{i_2} & \ldots & p_{i_s} \end{pmatrix}$, $s \geq 1$, consisting of a mapping from propositional variables to propositional formulas, and $A\sigma$ denotes the result of applying the substitution to formula A, which replaces each variable in A with its image under σ. This definition of substitution rule allows to use the simultaneous substitution of multiple formulas for multiple variables of A without any restrictions. The substitution rule is not sound.

If the depths of formulas φ_{i_j} $(1 \leq j \leq s)$ are restricted by some fixed d $(d \geq 0)$, then we have *d-restricted substitution* rule and we denote the corresponding system by $S^d\mathcal{F}$. 0-restricted substitution rule is named *renaming* rule.

We use also the well-known notions of proof, proof complexities and p-simulation given in [1]. The proof in any system Φ (Φ-proof) is a finite sequence of such formulas, each being an axiom of Φ, or is inferred from earlier formulas by one of the rules of Φ.

The total number of symbols, appearing in a formula φ, we call size of φ and denote by $|\varphi|$.

We define ℓ-*complexity* to be the *size* of a proof (= the total number of symbols) and t-*complexity* to be its *length* (= the total number of lines).

The minimal ℓ-complexity (t-complexity) of a formula φ in a proof system Φ we denote by ℓ_φ^Φ (t_φ^Φ).

Let Φ_1 and Φ_2 be two different proof systems.

Definition 1.. *The system Φ_2 p-ℓ-simulates Φ_1 ($\Phi_1 \prec_\ell \Phi_2$), if there exists a polynomial $p(\,)$ such, that for each formula φ, provable both in Φ_1 and Φ_2, we have $\ell_\varphi^{\Phi_2} \leq p\left(\ell_\varphi^{\Phi_1}\right)$.*

Definition 2.. *The system Φ_1 is p-ℓ-equivalent to system Φ_2 ($\Phi_1 \sim_\ell \Phi_2$), if Φ_1 and Φ_2 p-ℓ-simulate each other.*

Similarly p-t-simulation and p-t-equivalence are defined for t-complexity.

Definition 3.. *The system Φ_2 has exponential ℓ-speed-up (t-speed-up) over the system Φ_1, if there exists a sequence of such formulae φ_n, provable both in Φ_1 and Φ_2, that $\ell_{\varphi_n}^{\Phi_1} > 2^{\theta\left(\ell_{\varphi_n}^{\Phi_2}\right)}$ $\left(t_{\varphi_n}^{\Phi_1} > 2^{\theta\left(t_{\varphi_n}^{\Phi_2}\right)}\right)$.*

In this paper we compare under the p-simulation relation the proof systems $S\mathcal{F}$ and $S^d\mathcal{F}$ for some fixed integer $d \geq 0$, as well as the systems $S^{d_1}\mathcal{F}$ and $S^{d_2}\mathcal{F}$ for $d_1 \neq d_2$.

3. THE MAIN RESULT

The main result of our paper is the following statement

Theorem.

1. *For every fixed integer $d \geq 0$ $S^d\mathcal{F} \sim_\ell S\mathcal{F}$.*
2. *For every fixed integers $d_1 \geq 0$ and $d_2 \geq 0$ $S^{d_1}\mathcal{F} \sim_\ell \mathcal{S}^{\lceil \in}\mathcal{F}$ and for $d_1 \geq 1$ and $d_2 \geq 1$ $S^{d_1}\mathcal{F} \sim_\sqcup \mathcal{S}^{\lceil \in}\mathcal{F}$.*
3. *For every fixed integer $d \geq 1$ $S\mathcal{F}$ has exponential t-speed-up over the system $S^d\mathcal{F}$.*

The proof of the point 1. is based on the result of Buss, who proved that renaming Frege systems p-ℓ-simulate Frege systems with substitution without any restrictions [4]. By analogy is proved that $S^0\mathcal{F}$ p-ℓ-simulate $S^d\mathcal{F}$ for every $d \geq 0$.

The first statement of the point 2 follows from the statement of point 1. For proving the second statement of the point 2, it is not difficult to prove that for every $d \geq 1$ $S^d\mathcal{F} \prec_\sqcup \mathcal{S}^\infty\mathcal{F}$. Really, every d-restricted substitution can be realized step by step, using 1-restricted substitutions and renaming rule in case of need.

To prove the statement of point 3 we prove that for the formulas

$$\varphi_n = p_1 \supset (p_2 \supset (p_3 \supset \cdots \supset (p_n \supset p_1)\ldots)) \qquad n \geq 2$$

are true the following results:

$t_{\varphi_n}^{S\mathcal{F}} = O(\log_2 n)$ and $t_{\varphi_n}^{S^d\mathcal{F}} = \Omega(n)$ for every $d \geq 1$.

Remark. It is not difficult to see that the same results can be proved for some proof systems of non-classical propositional logic. In particular the statements of main theorem are true for Frege systems of Intuitionistic [5] and Minimal Logic [6].

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