Schauffler Theorem for Mediality

Ebrahim, Nazari
Payame Noor University
Gasersherin, Iran
e-mail: ebrahimnaravi@yahoo.com

ABSTRACT

In this paper we prove a necessary and sufficient condition for a full binary invertible algebra to satisfy the $\forall\exists(\forall)$-identity of mediality. A similar result is valid for full n-ary invertible algebras satisfying the $\forall\exists(\forall)$-identity of n-ary mediality.

Keywords

quasigroup operation, invertible algebra, full invertible algebra, mediality.

1. INTRODUCTION

A general concept of the $\forall\exists(\forall)$-identity is given in [1,2,3] (also see [4,5]). Let Q be a non empty set. The mapping $A : Q^2 \to Q$ is called a binary quasigroup operation if the equations $A(a, x) = b$ and $A(y, a) = b$ have unique solutions in the set Q for any $a, b \in Q$. Let L_Q be a set of all the binary quasigroup operations on the set Q. The algebra (Q, L_Q) is called a binary full invertible algebra. In paper [6] (also see [7,8]) a necessary and sufficient condition is given for a full binary invertible algebra, (Q, L_Q), to satisfy the $\forall\exists(\forall)$-identity of associativity:

$$\forall X, Y \exists X', Y' \forall x, y, z (X(Y(x, y), z) = X'(x, Y'(y, z)). \quad (1)$$

In paper [9] the algebraic proof of this result is given. A similar result for full ternary invertible algebras was proven in [10], and also for full n-ary invertible algebras in [11] (also see [12]). In paper [13] some stronger and more general results were proven.

These results, as well as the original result of Schauffler, are applicable in coding theory [14,15].

We need the following characterization of full binary invertible algebras of small order.

Theorem 1. If the cardinality of the set Q is less than, or equal to 3, then every quasigroup operation on the set Q is linear.

To prove this result the concept of group holomorphism [16] is necessary.

2. THE MAIN RESULT

Instead of equation (1), let us consider the following $\forall\exists(\forall)$-identity of mediality:

$$\forall X, Y, Z \exists X', Y', Z' \forall x, y, z, u (X(Y(x, y), Z(z, u)) =$$
$$X'(Y'(x, z), Z'(y, u)). \quad (2)$$

Theorem 2. In order the $\forall\exists(\forall)$-identity of mediality (2) to be held in a full binary invertible algebra (Q, L_Q), it is necessary and sufficient that the cardinality of the set Q is less than, or equal to 3.

A similar result is valid for full n-ary invertible algebras satisfying the $\forall\exists(\forall)$-identity of n-ary mediality.

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