Denial of Classical Ordering and Parallelism Axioms in Discrete Geometries' Space

Yuri, Grigoryan

European Regional Educational
Academy
Yerevan, Armenia
e-mail: piuesc@energinst.am

ABSTRACT

Using arithmetic graph coding methods, a definition of the Discrete Metric Space (DMS) is given. Three geometries, representing discrete analogs of Euclidean, non-Euclidean, and projective geometries, are axiomatically constructed.

Analysis of these geometries' characteristics shows that the two of the classical ordering axioms and the parallelism axiom are not applicable in the DMS.

Keywords

arithmetic graph, Euclidean, non-Euclidean, projective geometries.

1. INTRODUCTION

A discrete metric space, $\mathfrak{R}(D)$, based on the universal system of transformations was constructed by the author of the present paper in papers [1-5],. Three geometries, representing discrete analogies of Euclidean, non-Euclidean and projective geometries, were formulated in the above-said space.

The idea of construction of this space was founded on the concept of coding of discrete objects, set forth by the author in [6] and it is known as "arithmetic graph theory".

In this paper we show that two classical axioms of ordering and the parallelism axiom are invalid in the $\mathfrak{R}(D)$ space of discrete geometries.

For convenience of the reader, a number of definitions and theorems are taken from paper [4], which are reproduced below in a simpler form:

Definition 1. A set of real numbers except zero $\mathfrak{R}$ ($|\mathfrak{R}| \geq 3$) is called a "scalar set" if it satisfies two conditions:

(a) for any pair of different numbers $A, B \in \mathfrak{R}$:

$$A + B > 0,$$

(b) for any triad of different numbers $A, B, C \in \mathfrak{R}$:

$$AB + AC + BC > 0.$$

Note that $\mathfrak{R}$ sets are called "arithmetic sets" in paper [4]. Here we call it "a scalar set", which represents its content better.

Theorem 1. Each triad of real numbers $A, B, C \in \mathfrak{R}$ corresponds to a triangle $\Delta A_1B_1C_1$ having lengths of sides equal to:

$$a = |B_1C_1|,\ b = |A_1C_1|,\ c = |A_1B_1|, \tag{1}$$

where

$$A = \frac{b^2+c^2-a^2}{2},\ B = \frac{a^2+c^2-b^2}{2},\ C = \frac{a^2+b^2-c^2}{2}. \tag{2}$$

Corollary 1. A correspondence may be established between the Euclidean triangle $\Delta A_1B_1C_1$ and equal Euclidean triangle designated as ΔABC, where

$$a = |B_1C_1| = |BC| = \sqrt{B+C},\ b = |A_1C_1| = |AC| = \sqrt{A+C},\ c = |A_1B_1| = |AB| = \sqrt{A+B} \tag{3}$$

The area of ΔABC is obtained by Heron's relation, where values (3) are used:

$$S = \frac{1}{2}\sqrt{AB+AC+BC} \geq 0 \tag{4}$$

The major trigonometric functions are obtained through expressions (3) and (4):

$$\cos\angle BAC = \frac{A}{\sqrt{A^2+4S^2}},\ \sin\angle BAC = \frac{2S}{\sqrt{A^2+4S^2}} \tag{5}$$

$$\cos\angle ABC = \frac{B}{\sqrt{B^2+4S^2}},\ \sin\angle ABC = \frac{2S}{\sqrt{B^2+4S^2}} \tag{6}$$

$$\cos\angle ACB = \frac{C}{\sqrt{C^2+4S^2}},\ \sin\angle ACB = \frac{2S}{\sqrt{C^2+4S^2}} \tag{7}$$

$$tg\angle BAC = \frac{2S}{A},\ A = 2S\cdot ctg\angle BAC \tag{8}$$

$$tg\angle ABC = \frac{2S}{B},\ B = 2S\cdot ctg\angle ABC \tag{9}$$

$$tg\angle ACB = \frac{2S}{C},\ C = 2S\cdot ctg\angle ACB \tag{10}$$

The representation of these functions in the form of relations (5)-(10) depending both on the triangle area S and the triangle scalar parameters is assumed as a basis of a special trigonometry useful for further analysis of various geometries.

Since the further system of axioms is based on integers, we shall consider below only integer sets.

Let us fix an integer $D \leq -2$ and consider an infinite parametric set of integers:

$$\mathfrak{R}(D) = \{D,-D+1\} \cup \{D^2-D+i\},\ i = 0,1,2\ldots \tag{11}$$

Theorem 2. The parametric set of integers in (11) represents a scalar set for any $D \leq -2$.

The proof of this theorem is given in papers [2, 4]. One should observe that the set $\mathfrak{R}(D)$ includes the set of integers, $(-\infty,+\infty)$, except the quartet $\{-1,0,1,2\}$.

Theorem 3. The parametric set of integers in (11) represents a metric space with the distance function:

$$r(A,B) = \begin{cases} \sqrt{A+B}, & A\neq B \\ 0, & A=B. \end{cases} \tag{12}$$

for any fixed $D \leq -2$. (This metric was first introduced by the present author in 1982, see [5]).

We use the same notation for the constructed metric space as for the initial set in (11), $\mathfrak{R}(D)$, and call this space *the Discrete Geometries' Space (DGS)*. Note that since $0 \notin \mathfrak{R}(D)$, rectangular triangles do not exist in $\mathfrak{R}(D)$.

The graph having its vertices made distinguishable by some or other marks is called a "marked graph" in the sequel.

According to the Corollary 1, each Euclidean triangle $\Delta A_1B_1C_1$ marked by letters A_1, B_1 and C_1 corresponds to a marked ΔABC equal to the initial one, and in the second one the scalar values *A, B* and *C* are the markers given by the relation (2). On the other hand, the $\mathfrak{R}(D)$ space includes the set of all possible Euclidean triangles (A,B,C) with vertices marked by integers $A, B, C \in \mathfrak{R}(D)$ and having lengths of sides coincide with distances in $\mathfrak{R}(D)$, i.e.

$$|A;B| = r(A,B) = \sqrt{A+B}, |A;C| = r(A,C) = \sqrt{A+C},$$
$$|B;C| = r(B,C) = \sqrt{B+C},$$

In other words, under such mapping the marked triangle ΔABC represents an image of infinite number of equal-area triangles from the Euclidean space with specified side lengths of the initial triangle: $|A_1B_1| = c$, $|A_1C_1| = b$, and $|B_1C_1| = a$.

In the integer coding of vertices of geometric figures (graphs) – which follows – we use unequal integers from the set $\mathfrak{R}(D)$ for coding.

Definition 4. Any integer $\mathfrak{R}(D_0)$ is called a "point" of the Discrete Geometries Space from relation (11) for a fixed value of $D_0 \le -2$.

Definition 5. A pair of integers (X,Y) with $X, Y \in \mathfrak{R}(D_0)$ is called a "discrete line" for a fixed $D_0 \le -2$ if:

(1) $\sqrt{X+Y}$ is an integer, (13)

(2) there exists an integer $Z \in \mathfrak{R}(D_0)$ at which $XZ + YZ + XY = 0$.

For example, $Z = 5$ at $D_0 = -4$ and $(X,Y) = (20,-4)$.

Definition 6. A pair of positive integers (U,V) with $U,V \in \mathfrak{R}(D_0)$ is called a "discrete pseudo-straight line" for a fixed $D_0 \le -2$ if:

(1) $\sqrt{U+V}$ is an integer, (14)

(2) there exists a positive integer $T \in \mathfrak{R}(D_0)$ for which $UT + VT - UV = 0$.

For example, for $D_0 = -3$, three discrete pseudo-straight lines exist, $(U,V) = (28,21)$, $(U,V) = (48,16)$, and $(X,Y) = (156,13)$, if $T = 12$.

Definition 7. The discrete line (K,D_0), $K, D_0 \in \mathfrak{R}(D_0)$ is called a "basal discrete straight line" for a fixed $D_0 \le -2$ if:

(1) $K + D_0 = 1$,

(2) there exists a positive integer $C \in \mathfrak{R}(D_0)$ for which $KD_0 + CD_0 + CK = 0$.

For example, for $D_0 = -4$, one has $(K,D_0) = (5,-4)$ and $C = 20 \in \mathfrak{R}(D_0)$.

Definition 8. The discrete pseudo-straight line (A,B) is called a "basal discrete pseudo-line" for a fixed $D_0 \le -2$ if the following conditions are satisfied:

(1) $A + B = (1-2D_0)^2$;

(2) there exists a positive integer $C \in \mathfrak{R}(D_0)$ for which $AC + BC - AB = 0$.

For example, for $D_0 = -5$, one has $(A,B) = (66,55)$ and $C = 30 \in \mathfrak{R}(-5)$.

The concept of three different discrete planes $\mathscr{E}, \bar{\mathscr{E}}$ and $\mathscr{P}$ was introduced in paper [4] for any fixed $D_0 \in \mathfrak{R}(D)$ ($D_0 \le -2$) determined by the points:

$$\mathscr{E}\{A,B,K,D_0\},\ \bar{\mathscr{E}}\{A,B,K,D_0,C\},\ \mathscr{P}\{A,B,K,D_0,C,E,F\},$$

respectively called "a discrete plane", "a discrete extended plane" and "a limiting discrete plane". These planes may schematically be represented by a system of six equations with seven unknowns:

$$\left.\left.\left.\begin{array}{l} A + D_0 = B + K \\ AD_0 + BK = 0 \\ K + D_0 = 1 \end{array}\right] \mathscr{E} \atop KC + D_0C + KD_0 = 0 \right] \bar{\mathscr{E}} \atop \begin{array}{l} K^2 + D_0^2 + C^2 = E^2 \\ A^2 + B^2 + C^2 = F^2 \end{array} \right] \mathscr{P} \qquad (15)$$

For a fixed of $D_0 \le -2$, each sub-system $\{\mathscr{E}, \bar{\mathscr{E}}, \mathscr{P}\}$ of the system (15) has a unique integer solution (16) expressed through an odd parameter $n = 1 - 2D_0$ $(n \ge 5)$:

$$D_0 = -\frac{n-1}{2},\ K = \frac{n+1}{2},\ C = \frac{n^2-1}{4},$$
$$E = \frac{n^2+3}{4},\ B = \frac{n(n-1)}{2},\ A = \frac{n(n+1)}{2},\ F = \frac{3n^2+1}{4}. \qquad (16)$$

This system satisfies the condition

$$\mathscr{E} \supset \bar{\mathscr{E}} \supset \mathscr{P} \qquad (17)$$

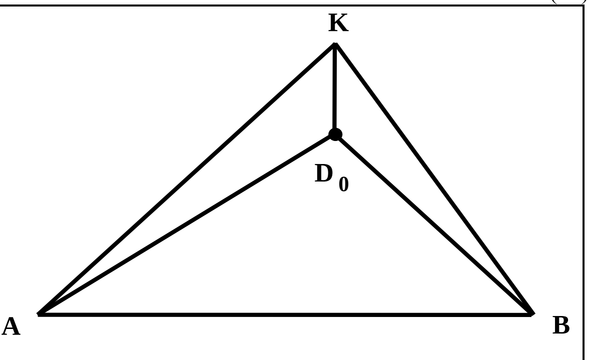

Fig. 1

It is to be noted that the odd parameter $n = 1 - 2D_0 \geq 5$, having the values: $5, 7, 9, 11, \ldots$ "quantizes" the whole space $\mathfrak{R}(D)$ in (11) into an infinite number of subspaces $\mathfrak{R}(D_0)$ satisfying system (15). The solution of the three systems of equations from (15) provides the existence of three different geometries $\mathscr{E}, \overline{\mathscr{E}}$ and $\mathscr{P}$, which we further call: *Discrete Euclidean Geometry (DEG), Discrete Non-Euclidean Geometry (DNEG), and Discrete Projective Geometry (DPG),* respectively.

Fig. 1 gives the schematic structure of DEG $\mathscr{E}\{A,B,K,D_0\}$.

Using relations (16) and distance expression (12) one can find the values of the six sides:

$$|K;D_0| = 1, |A;K| = \sqrt{2}\,\frac{n+1}{2}, |B;D_0| = \sqrt{2}\,\frac{n-1}{2},$$

$$|B;K| = |A;D_0| = \sqrt{2}\,\frac{\sqrt{n^2+1}}{2}, |A;B| = n$$

and the twelve angles:

$$\angle ABD_0 = \angle BAK = \angle AKD_0 = \frac{\pi}{4},\ \angle BD_0K = \frac{3\pi}{4},$$

$$\angle BKD_0 = \angle BAD_0 = arctg\,\frac{n-1}{n+1},$$

$$\angle D_0AK = \angle D_0BK = arctg\,\frac{1}{n},$$

$$\angle AKB = arctg\ n,\ \angle ABK = arctg\,\frac{n+1}{n-1},$$

$$\angle AD_0B = arctg\,(-n),\ \angle AD_0K = arctg\left(-\frac{n+1}{n-1}\right).$$

Let us construct now the arithmetic model of DEG $\mathscr{E}\{A,B,K,D_0\}$ in the Cartesian coordinate system. To do this we select four points on the Euclidean plane (Fig. 2):

$$(0,0),\ (1,0),\ \left(-\frac{n_0-1}{2},\frac{n_0+1}{2}\right),\ \left(-\frac{n_0-1}{2},-\frac{n_0-1}{2}\right).$$

Here $n_0 \geq 5$ is a fixed odd integer.

We code the selected points in the following way:

$$D_0(0,0),\ K(1,0),\ A\left(-\frac{n_0-1}{2},\frac{n_0+1}{2}\right),\ B\left(-\frac{n_0-1}{2},-\frac{n_0-1}{2}\right) \quad (18)$$

Using the Euclidean distance expression:

$$d(x,y) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

and metrics (12) we find the lengths of all possible six sides $|D_0;K|,|D_0;A|,|D_0;B|,|A;K|,|B;K|,|A;B|$ (Fig. 2):

$$d(D_0,K) = r(D_0,K) = 1,\ d(D_0,A) = r(D_0,A) = \sqrt{2}\,\frac{\sqrt{n_0^2+1}}{2},$$

$$d(D_0,B) = r(D_0,B) = \sqrt{2}\,\frac{n_0-1}{2},\ d(A,K) = r(A,K) = \sqrt{2}\,\frac{n_0+1}{2},$$

$$d(B,K) = r(B,K) = \sqrt{2}\,\frac{\sqrt{n_0^2+1}}{2},\ d(A,B) = r(A,B) = n_0. \quad (19)$$

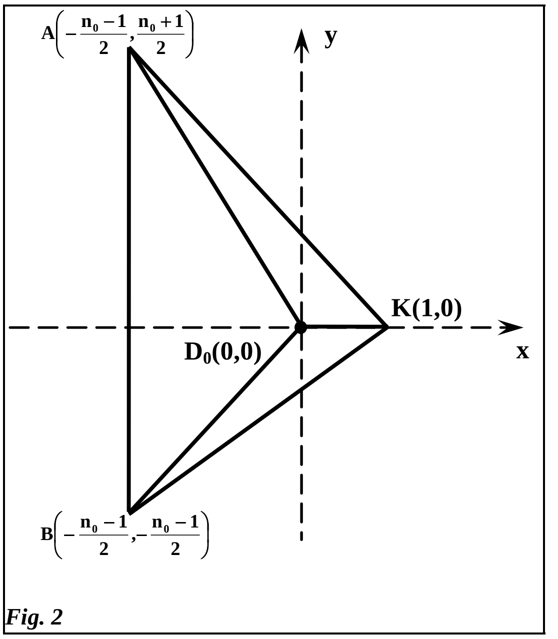

Fig. 2

System (19) shows that four points (18) having the structure shown in Fig. 2 serve an arithmetic model of the Discrete Euclidean Geometry $\mathscr{E}\{A,B,K,D_0\}$; OED.

Now we want to show that DEG refutes the Euclidean order axiom.

Theorem 4. The conditions of existence of the Discrete Euclidean Geometry, $\mathscr{E}\{A,B,K,D_0\}$:

$$\left.\begin{array}{l} A + D_0 = B + K \\ AD_0 + BK = 0 \\ K + D_0 = 1 \end{array}\right]$$

given in (15) refute the classical Euclidean order axiom.

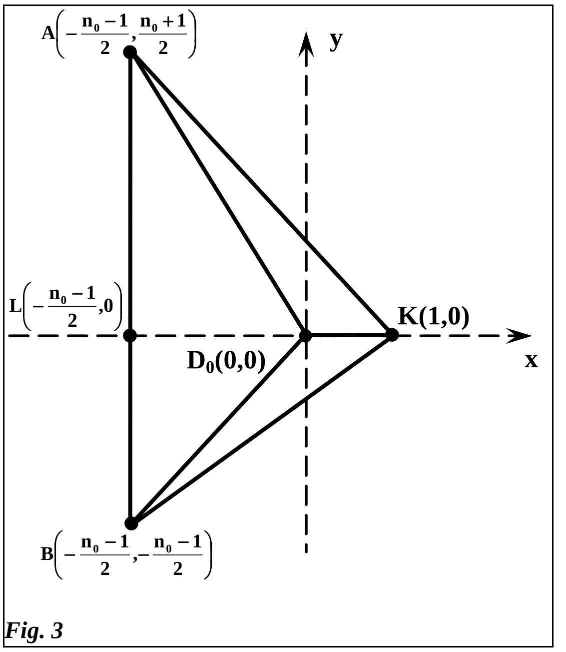

Fig. 3

Proof. Let the DEG $\mathscr{E}$ does not refute the order axiom. Then, taking into account the arithmetic model shown in Fig. 2 and due to existence of points (18) on the DEG plane there the point $L\left(-\frac{n_0-1}{2},0\right)$ must exist on the X-axis (Fig. 3), for which the following equality holds:

$$|L;D_0|=\frac{n_0-1}{2},|D_0;K|=1,|L;K|=\frac{n_0+1}{2}. \qquad (20)$$

Using the relation (1) and Fig. 3, let us map the points $\{K,D_0,L\}$ into the space $\mathfrak{R}(D_0)$. We obtain:

$$D_0=\frac{|L;D_0|^2+|D_0;K|^2-|L;K|^2}{2}=-\frac{n_0-1}{2}=D_0, \qquad (21)$$

$$K=\frac{|D_0;K|^2+|L;K|^2-|L;D_0|^2}{2}=\frac{n_0+1}{2}=K, \qquad (22)$$

$$L=\frac{|L;D_0|^2+|L;K|^2-|K;D_0|^2}{2}=\frac{n_0^2-1}{4}=C. \qquad (23)$$

It follows from the relations (21) and (22) that the points D_0 and K are mapped on themselves (i.e. remain intact), while the point L, according to (16) and (23), becomes:

$$L=C. \qquad (24)$$

In the same way as was done for (20), we have for the side length $|A;B|$:

$$|A;B|=n_0,|A;L|=\frac{n_0+1}{2},|B;L|=\frac{n_0-1}{2}. \qquad (25)$$

Using relations (1) and Fig. 3, let us map the points $\{A,B,L\}$ into the space $\mathfrak{R}(D_0)$. We obtain:

$$A=\frac{|A;B|^2+|A;L|^2-|L;B|^2}{2}=\frac{n_0(n_0+1)}{2}=A, \qquad (26)$$

$$B=\frac{|A;B|^2+|B;L|^2-|A;L|^2}{2}=\frac{n_0(n_0-1)}{2}=B, \qquad (27)$$

$$L=\frac{|A;L|^2+|B;L|^2-|A;B|^2}{2}=-\frac{n_0^2-1}{2}=-C. \qquad (28)$$

It follows from (26) and (27) that the points A and B are mapped on themselves (i.e. remain intact), while the point L, according to (16) and (28), becomes:

$$L=-C. \qquad (29)$$

Comparing the results (24) and (29) we obtain

$$C=-C\Rightarrow C=0,$$

which is impossible, since $0\notin\mathfrak{R}(D_0)$. Thus the initial assumption was false and the theorem is proved.

The proposed PDG $\mathfrak{R}(D_0)$ theory also refutes the classical parallelism axiom. Indeed, taking into account the triangle area expression (6), tangents (8)-(10) and the condition $KC+D_0C+KD_0=0$ from system (15), we obtain that $S_{\Delta KD_0C}=0$, whence it follows:

$$\angle K=\angle D_0=\angle C=0 .$$

Therefore, the KD_0C triangle represents a Lobachevsky triangle having zero sum of angles. Thus the parallelism axiom does not hold in the Projective Discrete Geometry $\mathfrak{R}(D_0)$.

The obtained results of invalidity of classical axioms of order and parallelism in $\mathfrak{R}(D)$ create a new view on the space and its properties.

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