Numerical Invariants for the Strong Product of Generalized Cycles

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ABSTRACT

In this paper the clique covering and the stable set numbers are investigated for the strong product of two generalized cycles. Methods are given to construct a minimum clique cover and a maximum stable set in case some conditions hold. A lower bound is given for the stable set number of the strong product of generalized cycles. The results here generalize some facts known for odd cycles.

1. INTRODUCTION

The investigation of stable set number of product graphs has come from the problem in information theory due to Shannon [1, 2]. Ore in [3] raised the following problem: Given a finite graph G, what are the necessary and sufficient conditions on G in order that $\alpha(G \times H) = \alpha(G) \times \alpha(H)$, for every finite graph H, where $\alpha(G)$ is the stable set number of G.

For the equality above a sufficient condition is found by Shannon [1]. Then Rosenfeld [4] proved its being not necessary and gave a necessary and sufficient condition, thereby introducing an invariant called ρ, the Rosenfeld number.

In [5], Hales obtained the non-multiplicative behavior of the clique covering and stable set numbers on the strong product of odd cycles. This work is closely related to it. The strong product of two generalized cycles is investigated. The following upper and lower bounds are known for the stable set number and clique covering number of product graphs [4, 5]:

$$\alpha(G \times H) \leq \min(\rho(G) \times \alpha(H), \alpha(G) \times \rho(H)),$$

$$\sigma(G \times H) \geq \max(\rho(G) \times \sigma(H), \sigma(G) \times \rho(H)),$$

where $\rho(G)$ is the Rosenfeld number of graph G. In case some conditions hold, methods are given to construct maximum stable set and minimum clique cover in the product of generalized cycles to achieve the bounds above.

2. PRELIMINARIES

A set of vertices of a graph is stable if no two vertices in it are adjacent. A stable set containing k vertices is called k-stable set. Let's denote by $\alpha(G)$-the number of vertices in a maximum stable set of G. A set of vertices of a graph is a clique if every two distinct vertices in it are adjacent and if it's maximal with respect to this property. A collection C of cliques is a clique-cover of graph G if $\bigcup_{Q \in C} Q = V(G)$, where $V(G)$ is the set of vertices of G. The clique-covering number of G, $\sigma(G)$, is the number of cliques in a minimum clique-cover of G. A graph is called k-regular if the degree of each vertex is k.

Generalized cycles are defined as follows:
Let's denote by C_n^k the $2k$-regular graph with n vertices which can be ordered on a circle so that each vertex is adjacent to the k vertices coming after and before it on the circle ($n > 2; 1 \leq k \leq [\frac{n-1}{2}]$).

For $c \in R$ real number we shall use the following notations:
$[c]$ - greatest integer less than or equal to c,
$]c[$ - least integer greater than or equal to c.

Also $\alpha_{mp} = \alpha(C_m^p), \sigma_{mp} = \sigma(C_m^p)$ and $r_{mp} = m mod(p+1)$.

The strong product of G_1 and G_2 is a graph G with vertices $V(G)$ and edges $E(G)$, where $V(G) = V(G_1) \times V(G_2)$ and $[(u_1, u_2), (v_1, v_2)] \in E(G)$ if and only if:

1. $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$, or

2. $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$, or

3. $(u_1, v_1) \in E(G_1)$ and $(u_2, v_2) \in E(G_2)$.

A non-negative real-valued function f on $V(G)$ is called admissible if for each clique C, $\sum_{v \in C} f(v) \leq 1$.

The Rosenfeld number $\rho(G)$ of a graph G is defined as [4, 5]:
$\rho(G) = \max_f \sum_{v \in V(G)} f(v)$, running over all f admissible functions.

One can deduce that, $\rho(C_n^k) = n/(k+1)$, while $\alpha(C_n^k) = [n/(k+1)]$ and $\sigma(C_n^k) =]n/(k+1)[$. The following inequalities are known for each graphs G and H [4, 5]

$$\alpha(G) \times \alpha(H) \leq \alpha(G \times H) \leq \min(\rho(G) \times \alpha(H), \alpha(G) \times \rho(H)),$$

$$\max(\rho(G) \times \sigma(H), \sigma(G) \times \rho(H)) \leq \sigma(G \times H) \leq \sigma(G) \times \sigma(H).$$

Hales [5] obtained the following results for the stable set and clique covering numbers of strong product of two odd cycles ($2 \leq k \leq n$):

$$\alpha(C_{2n+1} \times C_{2k+1}) = [\rho(C_{2n+1}) \times \alpha(C_{2k+1})],$$

$$\sigma(C_{2n+1} \times C_{2k+1}) =]\rho(C_{2n+1}) \times \sigma(C_{2k+1})[.$$

For the related results on the stable set number of products of cycles refer to [6, 7, 8].

3. THE STABLE SET NUMBER OF THE PRODUCT OF TWO GENERALIZED CYCLES

It can be deduced from the inequalities above that for generalized cycles we have:

$$\alpha(C_m^p \times C_n^k) \leq \alpha_{mp}\alpha_{nk} + \min([r_{mp}\alpha_{nk}/(p+1)], [r_{nk}\alpha_{mp}/(k+1)]),$$

and

$$\alpha(C_m^p \times C_n^k) \geq \alpha(C_m^p) \times \alpha(C_n^k) = \alpha_{mp}\alpha_{nk}.$$

One can notice that if $r_{mp} = 0$ or $r_{nk} = 0$ then,

$$\alpha(C_m^p \times C_n^k) = \alpha(C_m^p) \times \alpha(C_n^k).$$

The theorem below suggests a stronger lower bound:

Theorem 1. *If* $\alpha_{mp} r_{nk} \geq [\alpha_{nk}/(p+1)](k+1) r_{mp}$, *then*

$$\alpha(C_m^p \times C_n^k) \geq \alpha_{mp}\alpha_{nk} + [\alpha_{nk}/(p+1)] r_{mp}.$$

Proof. To prove the theorem it's enough to construct a stable set in the product graph. We shall construct $t = \alpha_{mp}$, α_{nk}-stable sets $S_0, \ldots, S_{t-1}$ in C_n^k graph, then shall decompose each of them into $p+1$ parts. Afterwards by constructing more r_{mp} stable sets in C_n^k we shall have m stable sets, $P_0, P_1, \ldots, P_{m-1}$. Finally, we shall show that the required stable set in the product graph is the following:

$$S = \bigcup_{i=0}^{m-1} B_i, B_i = \{(i, v)/v \in P_i\}.$$

Let's try to decompose α_{nk} number into $p+1$ almost equal parts. It will be used to decompose S_i sets. Let

$v = \alpha_{nk} mod(p+1)$, then

$\alpha_{nk} = v]\alpha_{nk}/(p+1)[+(p+1-v)[\alpha_{nk}/(p+1)].$

Let's define also a_i numbers according to the equality above,

$a_i =]\alpha_{nk}/(p+1)[, i = 0, \ldots, v-1;$

$a_i = [\alpha_{nk}/(p+1)], i = v, \ldots, p.$

Clearly in that case, $\alpha_{nk} = \sum_{i=0}^{p} a_i$. Suppose l is the minimum non-negative integer satisfying the inequality,

$$(l+1) r_{nk} \geq [\alpha_{nk}/(p+1)](k+1) r_{mp},$$

according to the supposition of the theorem $l < \alpha_{mp}$. Consider the following α_{nk}-stable sets in C_n^k graph,

$S_0 = \{0, (k+1), 2(k+1), \ldots, (\alpha_{nk}-1)(k+1)\},$

$S_1 = \{-r_{nk}, (k+1) - r_{nk}, 2(k+1) - r_{nk}, \ldots, (\alpha_{nk}-1)(k+1) - r_{nk}\},$

$S_2 = \{-2r_{nk}, (k+1) - 2r_{nk}, 2(k+1) - 2r_{nk}, \ldots, (\alpha_{nk}-1)(k+1) - 2r_{nk}\},$

...

$S_l = \{-lr_{nk}, (k+1) - lr_{nk}, 2(k+1) - lr_{nk}, \ldots, (\alpha_{nk}-1)(k+1) - lr_{nk}\},$

...

$S_{t-1} = \{-lr_{nk}, (k+1) - lr_{nk}, 2(k+1) - lr_{nk}, \ldots, (\alpha_{nk}-1)(k+1) - lr_{nk}\},$

$R = \{-(l+1)r_{nk}, (k+1) - (l+1)r_{nk}, 2(k+1) - (l+1)r_{nk}, \ldots, (\alpha_{nk}-1)(k+1) - (l+1)r_{nk}\}.$

Operations here are considered to be done by $modn$. Consider the elements of sets $S_0, \ldots, S_{t-1}$ in the specified order and split each of them into $p+1$ parts (so that i-th set cardinality is a_i). We shall get $P_0, P_1, \ldots, P_{(p+1)t-1}$ sets. Now let's consider the elements of R in the specified order and separate from them the first r_{mp} sets with cardinality $[\alpha_{nk}/(p+1)]$. We shall get $P_0, P_1, \ldots, P_{m-1}$ stable sets in C_n^k graph.

To finalize the proof of the theorem it remains to show that the constructed set S is a stable set in the product graph. It suffices to show that each sequential $p+1$ sets in the cyclic sequence of $P_0, P_1, \ldots, P_{m-1}$ sets are pair-wise disjoint and the union of the $p+1$ sets is a stable set in C_n^k. Consider any such sequence of sets $P_{imod(n)}, P_{(i+1)mod(n)}, \ldots, P_{(i+p)mod(n)}$. If P_0 and P_{m-1} aren't present in the sequence at the same time, then the statement is true according to the construction, otherwise the statement implies from the definition of number l above. □

Corollary 1. *For every* C_m^p *and* C_n^k *generalized cycles holds,*

$$\alpha(C_m^p \times C_n^k) \geq \alpha_{mp}\alpha_{nk} + \min([\alpha_{nk}/(p+1)] r_{mp}, [\alpha_{mp}/(k+1)] r_{nk}),$$

particularly,

$$\alpha(C_n^k \times C_n^k) \geq \alpha_{nk}^2 + [\alpha_{nk}/(k+1)] r_{nk}.$$

Proof. Clearly, it suffices to prove only the first inequality. If the condition of Theorem 1 is satisfied $\alpha_{mp} r_{nk} \geq [\alpha_{nk}/(p+1)](k+1) r_{mp}$, then the proof of corollary is immediate, otherwise we have

$$\alpha_{mp} r_{nk} < [\alpha_{nk}/(p+1)](k+1) r_{mp},$$

hence

$$\alpha_{nk} r_{mp} \geq [\alpha_{mp}/(k+1)](p+1) r_{nk},$$

applying Theorem 1 we get

$\alpha(C_m^p \times C_n^k) \geq \alpha_{mp}\alpha_{nk} + [\alpha_{mp}/(k+1)] r_{nk} \geq \alpha_{mp}\alpha_{nk} + \min([\alpha_{nk}/(p+1)] r_{mp}, [\alpha_{mp}/(k+1)] r_{nk})$

and the corollary is proved. □

Corollary 2. *Consider the graph* C_n^k *and the cycle* C_{2h+1}. *If* $r_{nk} h \geq [\alpha_{nk}/2](k+1)$, *then*

$\alpha(C_{2h+1} \times C_n^k) = \alpha(C_{2h+1}) \times \alpha_{nk} + [\alpha_{nk}/2] = [\rho(C_{2h+1}) \times \alpha_{nk}].$

Proof. If we take into account that $C_{2h+1} = C_m^p$ for $m = 2h+1$ and $p = 1$, then we can apply Theorem 1:

$\alpha(C_m^p \times C_n^k) \geq \alpha_{mp}\alpha_{nk} + [\alpha_{nk}/2],$

but from the other side we have:

$$\alpha(C^p_m \times C^k_n) \leq [\rho(C^p_m) \times \alpha(C^k_n)] = \alpha_{mp}\alpha_{nk} + [\alpha_{nk}/2],$$

therefore the corollary is proved. □

4. THE CLIQUE COVERING NUMBER OF THE PRODUCT OF TWO GENERALIZED CYCLES

Theorem 2. Let C^p_m and C^k_n be generalized cycles. If the following conditions hold:

1)$p+1 \geq 2r_{mp}, r_{mp} \neq 0, r_{nk} \neq 0,$

2)$(\sigma(C^p_m)-1)(k+1-r_{nk}) \leq [\sigma(C^k_n)/2](k+1)$, *then*

$$\sigma(C^p_m \times C^k_n) =]\sigma(C^p_m) \times \rho(C^k_n)[= \sigma(C^p_m) \times \sigma(C^k_n) - [\sigma(C^p_m)\frac{k+1-r_{nk}}{k+1}]. \quad (1)$$

Proof. Since the right hand is a lower bound for σ it's enough to construct a clique cover to attain that bound. Let's denote the vertices of C^p_m and C^k_n by numbers $0, 1, ..., m-1$ and $0, 1, ..., n-1$ correspondingly. Then for the vertex $(x, y) \in V(C^p_m \times C^k_n)$ let

$$Q(x, y) = \{(x+i, y+j) : i = 0, ..., p; j = 0, ..., k\}$$

be a clique in the product graph $C^p_m \times C^k_n$ ($x+i$ and $y+j$ are taken by modulo m and n respectively). Let's denote $t = [\sigma(C^k_n)/2]$.

Consider the following families of cliques

$Q^0_0 = \{Q(0, (k+1)i) : i = 0, ..., t-1\},$

$Q^1_0 = \{Q(r_{mp}, (k+1)i) : i = t, ..., \sigma_{nk}-1\},$

$Q^0_1 = \{Q(p+1, k+1-r_{nk}+(k+1)i) : i = 0, ..., t-1\},$

$Q^1_1 = \{Q(p+1+r_{mp}, k+1-r_{nk}+(k+1)i) : i = t, ..., \sigma_{nk}-1\},$

...

$Q^0_{\sigma_{mp}-2} = \{Q((p+1)(\sigma_{mp}-2), (k+1-r_{nk})(\sigma_{mp}-2)+(k+1)i) : i = 0, ..., t-1\},$

$Q^1_{\sigma_{mp}-2} = \{Q((p+1)(\sigma_{mp}-2)+r_{mp}, (k+1-r_{nk})(\sigma_{mp}-2)+(k+1)i) : i = t, ..., \sigma_{nk}-1\},$

$Q_{\sigma_{mp}-1} = \{Q((p+1)(\sigma_{mp}-1), (k+1-r_{nk})(\sigma_{mp}-1)+(k+1)i) : i = 0, ..., \sigma_{nk}-[\sigma_{mp}\frac{k+1-r_{nk}}{k+1}]-1\}.$

We will show that the union of those families above is a clique cover for the product graph. Obviously its cardinal number is equal to the right hand of the equality (1). Let $(x, y) \in V(C^p_m \times C^k_n)$, then the following 3 cases are possible

1)$r_{mp} \leq x \leq m-r_{mp}-1 = (\sigma_{mp}-1)(p+1)-1.$
Then $x = s(p+1)+c$, $0 \leq s \leq \sigma_{mp}-2$, $0 \leq c \leq p$. If $s = 0$, then $r_{mp} \leq c \leq p$ and clearly $(x, y) \in Q^0_0 \cup Q^1_0$, otherwise $(x, y) \in Q^1_{s-1} \cup Q^0_s \cup Q^1_s$.

2)$m-r_{mp} \leq x \leq m.$
Then $x = s(p+1)+c$, $s = \sigma_{mp}-1$, $0 \leq c \leq r_{mp}-1$. If $0 \leq y \leq (k+1-r_{nk})(\sigma_{mp}-1)-1$, then $(x, y) \in Q^1_{\sigma_{mp}-2}$, otherwise we have

$$(k+1-r_{nk})(\sigma_{mp}-1)+(k+1)(\sigma_{nk}-[\sigma_{mp}\frac{k+1-r_{nk}}{k+1}]) \geq \quad (2)$$
$$\geq (k+1)\sigma_{nk}+(k+1-r_{nk})(\sigma_{mp}-1)-\sigma_{mp}(k+1-r_{nk}) \geq$$
$$\geq (k+1)\sigma_{nk}-(k+1-r_{nk}) \geq n,$$

therefore $(x, y) \in Q_{\sigma_{mp}-1}$.

3)$0 \leq x \leq r_{mp}-1.$
In this case if $0 \leq y \leq t(k+1)-1$, then $(x, y) \in Q^0_0$, otherwise we have the conditions of the theorem and the inequality (2). According to the 1st condition of the theorem, $Q_{\sigma_{mp}-1}$ covers a part of vertices with first coordinate up to $r_{mp}-1$(and more if $p+1$ is strictly greater than $2r_{mp}$). According to the 2nd condition of the theorem and inequality (2), there are no uncovered vertices with second coordinate $t(k+1) \leq y \leq n-1 (0 \leq x \leq r_{mp}-1)$, hence $(x, y) \in Q_{\sigma_{mp}-1}$.

Therefore the union of the mentioned families is a clique cover of the product graph with required cardinality. □

Corollary 3. Let C^p_m and C^k_n be generalized cycles. If $p+1 = 2r_{mp}$ and $k+1 = 2r_{nk}$, then

$$\sigma(C^p_m \times C^k_n) = max(]\sigma(C^p_m) \times \rho(C^k_n)[,]\sigma(C^k_n) \times \rho(C^p_m)[).$$

Proof. The right hand of the suggested equality is a lower bound for $\sigma(C^p_m \times C^k_n)$. If the 2nd condition of Theorem 2 holds then the proof is immediate, otherwise we have

$(\sigma(C^p_m)-1)(k+1-r_{nk}) > [\sigma(C^k_n)/2](k+1)$ and since $k+1 = 2r_{nk}$ we get

$\sigma(C^p_m)/2 > [\sigma(C^k_n)/2] + 1/2 \geq \sigma(C^k_n)/2,$

$[\sigma(C^p_m)/2] \geq \frac{\sigma(C^k_n)-1}{2}.$

The latter is the second condition of Theorem 2 and with $k+1 = 2r_{nk}$ equality it implies that

$\sigma(C^p_m \times C^k_n) =]\sigma(C^k_n) \times \rho(C^p_m)[\leq$
$max(]\sigma(C^p_m) \times \rho(C^k_n)[,]\sigma(C^k_n) \times \rho(C^p_m)[),$

hence

$$\sigma(C^p_m \times C^k_n) = max(]\sigma(C^p_m) \times \rho(C^k_n)[,]\sigma(C^k_n) \times \rho(C^p_m)[).$$

□

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