On interval total colorings of bipartite graphs

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ABSTRACT

An interval total $t-$coloring of a graph G is a total coloring of G with colors $1,2,\dots,t$ such that at least one vertex or edge of G is colored by color i, $i=1,2,\dots,t$, and the edges incident to each vertex $v\in V(G)$ together with v are colored by $d_G(v)+1$ consecutive colors, where $d_G(v)$ is the degree of the vertex v in G. In this paper interval total colorings of bipartite graphs are investigated.

Keywords
Total coloring, interval edge coloring, interval total coloring, bipartite graph.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v\in V(G)$ is denoted by $d_G(v)$, the maximum degree of a vertex of G - by $\Delta(G)$ and the chromatic index of G - by $\chi'(G)$. A proper edge coloring of a graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. If α is a proper edge coloring of G and $v\in V(G)$ then $S(v,\alpha)$ denotes the set of colors of edges incident to v. A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. The total chromatic number $\chi''(G)$ is the smallest number of colors needed for total coloring of G. If α is a total coloring of a graph G then $\alpha(v)$ and $\alpha(e)$ denote the color of a vertex $v\in V(G)$ and the color of an edge $e\in E(G)$ in the coloring α, respectively. For a total coloring α of a graph G and for any $v\in V(G)$ define the set $S[v,\alpha]$ as follows:

$$S[v,\alpha]\equiv\{\alpha(v)\}\cup\{\alpha(e)\mid e\text{ is incident to }v\}.$$

For two integers $a\le b$ the set $\{a,a+1,\dots,b\}$ is denoted by $[a,b]$.

An interval total $t-$coloring [5,6] of a graph G is a total coloring of G with colors $1,2,\dots,t$ such that at least one vertex or edge of G is colored by color i, $i=1,2,\dots,t$, and the edges incident to each vertex $v\in V(G)$ together with v are colored by $d_G(v)+1$ consecutive colors.

For $t\ge 1$ let $\mathfrak{T}_t$ denote the set of graphs which have an interval total $t-$coloring, and assume: $\mathfrak{T}\equiv\bigcup_{t\ge1}\mathfrak{T}_t$. For a graph $G\in\mathfrak{T}$ the least value of t, for which $G\in\mathfrak{T}_t$, is denoted by $w_\tau(G)$.

In this paper interval total colorings of bipartite graphs are investigated.

The terms and concepts that we do not define can be found in [10,11].

2. MAIN RESULTS

Lemma 1. For any $n\ge 2$, $P_n\in\mathfrak{T}$ and $w_\tau(P_n)=3$.

Proof. Let

$V(P_n)=\{v_1,v_2,\dots,v_n\}$ and $E(P_n)=\{(v_i,v_{i+1})\mid 1\le i\le n-1\}$.

Clearly, lemma is true for the case $n=2$.

Assume that $n\ge 3$.

Case 1: $n=3k$ or $n=3k+2$, $k\in N$.

Define a total coloring α of the graph P_n in the following way:

1. for $i=1,2,\dots,n$

$$\alpha(v_i)=\begin{cases}2,\text{ if } i\equiv 0(\mathrm{mod}\,3),\\1,\text{ if } i\equiv 1(\mathrm{mod}\,3),\\3,\text{ if } i\equiv 2(\mathrm{mod}\,3),\end{cases}$$

2. for $j=1,2,\dots,n-1$

$$\alpha((v_j,v_{j+1}))=\begin{cases}3,\text{ if } j\equiv 0(\mathrm{mod}\,3),\\2,\text{ if } j\equiv 1(\mathrm{mod}\,3),\\1,\text{ if } j\equiv 2(\mathrm{mod}\,3).\end{cases}$$

Case 2: $n=3k+1$, $k\in N$.

Define a total coloring α of the graph P_n in the following way:

1. for $i=1,2,\dots,n$

$$\alpha(v_i)=\begin{cases}3,\text{ if } i\equiv 0(\mathrm{mod}\,3),\\2,\text{ if } i\equiv 1(\mathrm{mod}\,3),\\1,\text{ if } i\equiv 2(\mathrm{mod}\,3),\end{cases}$$

2. for $j=1,2,\dots,n-1$

$$\alpha\left(\left(v_j,v_{j+1}\right)\right)=\begin{cases}1, \text{ if } j\equiv 0 (\bmod 3),\\ 3, \text{ if } j\equiv 1 (\bmod 3),\\ 2, \text{ if } j\equiv 2 (\bmod 3).\end{cases}$$

It is easy to see that α is an interval total $3-$coloring of the graph P_n and, therefore, $P_n \in \mathfrak{T}$ and $w_\tau\left(P_n\right)=3$.

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Lemma 2. For any $n\geq 3$, $C_n \in \mathfrak{T}$ and

$$w_\tau\left(C_n\right)=\begin{cases}3, \text{ if } n=3k,\ k\in N,\\ 4, \text{ otherwise.}\end{cases}$$

Proof. Let

$$V\left(C_n\right)=\left\{v_1,v_2,\ldots,v_n\right\} \text{ and}$$

$$E\left(C_n\right)=\left\{\left(v_i,v_{i+1}\right)\mid 1\leq i\leq n-1\right\}\cup\left\{\left(v_1,v_n\right)\right\}.$$

First of all, we prove that C_n has an interval total $3-$coloring, if $n=3k,\ k\in N$, and C_n has an interval total $4-$coloring, if $n\neq 3k,\ k\in N$. We distinguish three cases.

Case 1: $n=3k$, $k\in N$.

Define a total coloring α of the graph C_n as follows:

1. for $i=1,2,\ldots,n$

$$\alpha(v_i)=\begin{cases}2, \text{ if } i\equiv 0 (\bmod 3),\\ 1, \text{ if } i\equiv 1 (\bmod 3),\\ 3, \text{ if } i\equiv 2 (\bmod 3),\end{cases}$$

2. for $j=1,2,\ldots,n-1$

$$\alpha\left(\left(v_j,v_{j+1}\right)\right)=\begin{cases}3, \text{ if } j\equiv 0 (\bmod 3),\\ 2, \text{ if } j\equiv 1 (\bmod 3),\\ 1, \text{ if } j\equiv 2 (\bmod 3),\end{cases}$$

3. $\alpha\left(\left(v_1,v_n\right)\right)=3.$

Case 2: $n\neq 3k$, $k\in N$ and n is even.

Define a total coloring α of the graph C_n as follows:

1. for $i=1,2,\ldots,n$

$$\alpha(v_i)=\begin{cases}4, \text{ if } i\equiv 0 (\bmod 2),\\ 1, \text{ if } i\equiv 1 (\bmod 2),\end{cases}$$

2. for $j=1,2,\ldots,n-1$

$$\alpha\left(\left(v_j,v_{j+1}\right)\right)=\begin{cases}2, \text{ if } j\equiv 0 (\bmod 2),\\ 3, \text{ if } j\equiv 1 (\bmod 2),\end{cases}$$

3. $\alpha\left(\left(v_1,v_n\right)\right)=2.$

Case 3: $n\neq 3k$, $k\in N$ and n is odd.

Define a total coloring α of the graph C_n as follows:

1. for $i=1,2,\ldots,n$

$$\alpha(v_i)=\begin{cases}4, \text{ if } i\equiv 0 (\bmod 2),\ i\neq n-1,\\ 1, \text{ if } i\equiv 1 (\bmod 2),\ i\neq n,\\ 2, \text{ if } i=n-1,\\ 3, \text{ if } i=n,\end{cases}$$

2. for $j=1,2,\ldots,n-1$

$$\alpha\left(\left(v_j,v_{j+1}\right)\right)=\begin{cases}2, \text{ if } j\equiv 0 (\bmod 2),\ j\neq n-1,\\ 3, \text{ if } j\equiv 1 (\bmod 2),\\ 4, \text{ if } j=n-1,\end{cases}$$

3. $\alpha\left(\left(v_1,v_n\right)\right)=2.$

It is easy to check that α is an interval total $3-$coloring of the graph C_n, if $n=3k,\ k\in N$, and an interval total $4-$coloring of the graph C_n, if $n\neq 3k,\ k\in N$. Hence, for any $n\geq 3$, $C_n\in\mathfrak{T}$ and $w_\tau\left(C_n\right)\leq 3$, if $n=3k,\ k\in N$, and $w_\tau\left(C_n\right)\leq 4$, if $n\neq 3k,\ k\in N$. On the other hand, since

$$w_\tau\left(C_n\right)\geq \chi''\left(C_n\right)=\begin{cases}3, \text{ if } n=3k,\ k\in N,\\ 4, \text{ otherwise,}\end{cases} [11]$$

then $w_\tau\left(C_n\right)\geq 3$, if $n=3k,\ k\in N$, and $w_\tau\left(C_n\right)\geq 4$, if $n\neq 3k,\ k\in N$.

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Every connected component of a graph G with $\Delta(G)\leq 2$ is a path or a cycle, so from lemma 1 and 2 any component can be intervally colored with no more than 4 colors. Thus we have

Theorem 1. If G is a graph with $\Delta(G)\leq 2$ then $G\in\mathfrak{T}$ and $w_\tau\left(G\right)\leq 4$.

In [9] A.S. Shashikyan proved the following:

Theorem 2. If G is a bipartite graph with $\Delta(G)\leq 3$ then $G\in\mathfrak{T}$ and $w_\tau\left(G\right)\leq 5$.

Now we consider bipartite graphs with $\Delta(G)\leq 4$.

Theorem 3. If $G=\left(U,V,E\right)$ is a bipartite graph with $\Delta(G)\leq 4$ and G has a $2-$factor then $G\in\mathfrak{T}$ and $w_\tau\left(G\right)\leq 6$.

Proof. First of all, note that if $\Delta(G)\leq 3$ then this theorem follows from theorem 2.

Assume that $\Delta(G)=4$.

Let F denotes the $2-$factor of the graph G. Clearly, F consists of even cycles. The edges of these cycles can be colored alternately by 2 and 3. Consider the subgraph $G\setminus F$ of the graph G. Clearly, $G\setminus F$ is a bipartite graph and all its vertices have degree 1 or 2, therefore its components are paths or even cycles. The edges of these paths and cycles we color alternately by 1 and 4. Let α be an obtained edge coloring.

Now define a total coloring β of the graph G in the following way:

1. for every $u\in U$

$$\beta\left(u\right)=\begin{cases}1, \text{ if } S\left(u,\alpha\right)=\left[1,4\right] \text{ or } S\left(u,\alpha\right)=\left[1,3\right] \text{ or } S\left(u,\alpha\right)=\left[1,2\right],\\ 2, \text{ if } S\left(u,\alpha\right)=\left[2,4\right] \text{ or } S\left(u,\alpha\right)=\left[2,3\right],\\ 3, \text{ if } S\left(u,\alpha\right)=\left[3,4\right],\end{cases}$$

2. for every $v\in V$

$$\beta\left(v\right)=\begin{cases}4, \text{ if } S\left(v,\alpha\right)=\left[1,2\right],\\ 5, \text{ if } S\left(v,\alpha\right)=\left[1,3\right] \text{ or } S\left(v,\alpha\right)=\left[2,3\right],\\ 6, \text{ if } S\left(v,\alpha\right)=\left[1,4\right] \text{ or } S\left(v,\alpha\right)=\left[2,4\right] \text{ or } S\left(v,\alpha\right)=\left[3,4\right],\end{cases}$$

3. for every $e\in E(G)$ $\beta\left(e\right)=\alpha\left(e\right)+1$.

It is easy to see that β is an interval total $6-$coloring of the graph G and, therefore, $G\in\mathfrak{T}$ and $w_\tau\left(G\right)\leq 6$.

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Theorem 4. If $G=(U,V,E)$ is a bipartite graph with $\Delta(G)\le 4$ and no vertex of degree 3 then $G\in\mathfrak{T}$ and $w_\tau(G)\le 6$.

Proof. First of all, note that if $\Delta(G)\le 3$ then this theorem follows from theorem 2.

Assume that $\Delta(G)=4$.

From the results of [2] it follows that G has an interval edge $4-$coloring [1]. Let α be this edge coloring. For a graph G define an edge coloring β as follows: for every $e\in E(G)$ $\beta(e)=\alpha(e)+1$.

First, we color the vertices of G with $d_G(w)\ge 2$ ($w\in V(G)$) in the following way:

1. for every $u\in U$

$$\gamma(u)=\begin{cases}1, \text{ if } d_G(u)=4 \text{ or } S(u,\beta)=[2,3],\\ 2, \text{ if } S(u,\beta)=[3,4],\\ 3, \text{ if } S(u,\beta)=[4,5],\end{cases}$$

2. for every $v\in V$

$$\gamma(v)=\begin{cases}4, \text{ if } S(v,\beta)=[2,3],\\ 5, \text{ if } S(v,\beta)=[3,4],\\ 6, \text{ if } d_G(v)=4 \text{ or } S(v,\beta)=[4,5].\end{cases}$$

Next, we color the vertices of G with $d_G(w)\le 1$ ($w\in V(G)$) in the following way:

1. for every $u\in U$

$$\varphi(u)=\begin{cases}1, \text{ if } d_G(u)=0,\\ s,\ s\in\{\beta((u,v))-1,\beta((u,v))+1\}\setminus\gamma(v),\\ \qquad \text{if } d_G(u)=1, (u,v)\in E(G), \text{ where } d_G(v)\ge 2,\\ \beta((u,v))-1, \text{ if } d_G(u)=d_G(v)=1 \text{ and } (u,v)\in E(G),\end{cases}$$

2. for every $v\in V$

$$\varphi(v)=\begin{cases}4, \text{ if } d_G(v)=0,\\ t,\ t\in\{\beta((u,v))-1,\beta((u,v))+1\}\setminus\gamma(u),\\ \qquad \text{if } d_G(v)=1, (u,v)\in E(G), \text{ where } d_G(u)\ge 2,\\ \beta((u,v))+1, \text{ if } d_G(u)=d_G(v)=1 \text{ and } (u,v)\in E(G).\end{cases}$$

Finally, define a total coloring ψ of the graph G as follows:

1. for every $e\in E(G)$ $\psi(e)=\varphi(e)$,
2. for every $w\in V(G)$ ($d_G(w)\ge 2$) $\psi(w)=\gamma(w)$,
3. for every $w\in V(G)$ ($d_G(w)\le 1$) $\psi(w)=\varphi(w)$.

It is not difficult to see that ψ is an interval total coloring of the graph G with no more than 6 colors. Thus $G\in\mathfrak{T}$ and $w_\tau(G)\le 6$.

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Theorem 5. Let $G=(U,V,E)$ be a bipartite graph such that

1. $\forall u\in U$ $d_G(u)=r$ $(r\ge 2)$,
2. $\forall v\in V$ $r-1\le d_G(v)\le r$,

then $G\in\mathfrak{T}$ and $r+1\le w_\tau(G)\le r+2$.

Proof. Since G is a bipartite graph then $\chi'(G)=\Delta(G)=r$. Let α be a proper edge coloring of G with colors $2,3,\dots,r+1$. Clearly, $S(u,\alpha)=[2,r+1]$ for any $u\in U$.

Define a total coloring β of the graph G as follows:

1. for every $e\in E(G)$ $\beta(e)=\alpha(e)$,
2. for every $u\in U$ $\beta(u)=1$,
3. for every $v\in V$

$$\beta(v)=\begin{cases}s,\ s\in[2,r+1]\setminus S(v,\alpha) \text{ if } d_G(v)=r-1,\\ r+2, \text{ otherwise.}\end{cases}$$

It is easy to check that β is an interval total coloring of the graph G with no more than $r+2$ colors, hence $G\in\mathfrak{T}$ and $w_\tau(G)\le r+2$. On the other hand, clearly $w_\tau(G)\ge r+1$.

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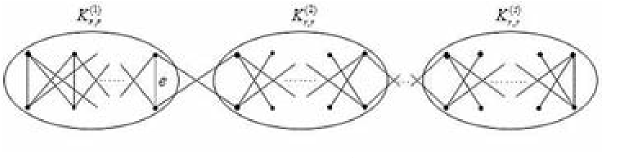
Corollary 1. Let G be an $r-$regular ($r\ge 2$) bipartite graph. Then $G\in\mathfrak{T}$ and $r+1\le w_\tau(G)\le r+2$.

Corollary 2. Let G be an $(r,r-1)-$biregular ($r\ge 2$) bipartite graph. Then $G\in\mathfrak{T}$ and $w_\tau(G)=r+1$.

Theorem 6. For any $r,s\ge 3$ there is an $r-$regular bipartite graph G such that $|V(G)|=2rs$, $G\in\mathfrak{T}$ and $w_\tau(G)=r+2$.

Proof. For the proof of the theorem it suffices to construct a necessary graph G. Take s copies of the complete bipartite graph $K_{r,r}$ and join their vertices as it shown in the figure below:



The graph G.

Clearly, G is an $r-$regular bipartite graph and $|V(G)|=2rs$. Now we show that G has no interval total $(r+1)-$coloring. Suppose, to the contrary, that α is an interval total $(r+1)-$coloring of G. It is easy to see that α induces a total $(r+1)-$coloring of the graph $K^{(1)}_{r,r}-e$, which contradicts the equality $\chi''(K^{(1)}_{r,r}-e)=r+2$ ($r\ge 3$) [11]. From this and corollary 1 we have $G\in\mathfrak{T}$ and $w_\tau(G)=r+2$.

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Remark. From corollary 1 and theorem 6 we have that $G\in\mathfrak{T}$, $w_\tau(G)=r+1$ or $w_\tau(G)=r+2$ for any $r-$regular bipartite graph G. In [8] it was proved that the problem of determining whether $\chi''(G)=r+1$ is $NP-$complete even for cubic bipartite graphs. Therefore we can conclude that verification whether $w_\tau(G)=r+1$ for an $r-$regular ($r\ge 3$) bipartite graph G is also $NP-$complete.

Theorem 7. Let G be an $(r,2)-$biregular ($r \geq 3$) bipartite graph. Then $G \in \mathfrak{T}$ and $r+1 \leq w_\tau(G) \leq r+2$.

Proof. Let G be an $(r,2)-$biregular ($r \geq 3$) bipartite graph with bipartition (U,V). Consider two cases.

Case 1: r is even.

From the results of [3,4] it follows that G has an interval edge $r-$coloring. Let α be this edge coloring. For a graph G define an edge coloring β as follows: for every $e \in E(G)$ $\beta(e) = \alpha(e)+1$.

Define a total coloring γ of the graph G as follows:

1. for every $e \in E(G)$ $\gamma(e) = \beta(e)$,
2. for every $u \in U$ $\gamma(u) = 1$,
3. for every $v \in V$

$$\gamma(v) = \begin{cases} \min S(v,\beta)-1, \text{ if } \min S(v,\beta) \geq 3, \\ \max S(v,\beta)+1, \text{ otherwise.} \end{cases}$$

It is easy to check that γ is an interval total $(r+1)-$coloring of the graph G, hence $G \in \mathfrak{T}$ and $w_\tau(G) = r+1$.

Case 2: r is odd.

From the results of [3,4] it follows that G has an interval edge $(r+1)-$coloring. Let α be this edge coloring. For a graph G define an edge coloring β as follows: for every $e \in E(G)$ $\beta(e) = \alpha(e)+1$.

Define a total coloring γ of the graph G as follows:

1. for every $e \in E(G)$ $\gamma(e) = \beta(e)$,
2. for every $u \in U$

$$\gamma(u) = \begin{cases} 1, \text{ if } S(u,\beta) = [2,r+1], \\ 2, \text{ otherwise.} \end{cases}$$

3. for every $v \in V$

$$\gamma(v) = \begin{cases} \min S(v,\beta)-1, \text{ if } \min S(v,\beta) \geq 4, \\ \max S(v,\beta)+1, \text{ otherwise.} \end{cases}$$

It is easy to check that γ is an interval total $(r+2)-$coloring of the graph G, hence $G \in \mathfrak{T}$ and $w_\tau(G) \leq r+2$.

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For trees P.A. Petrosyan and A.S. Shashikyan proved the following:

Theorem 8. [7] If T is a tree then $T \in \mathfrak{T}$ and $w_\tau(T) \leq \Delta(T)+2$.

For complete bipartite graphs P.A. Petrosyan proved the following:

Theorem 9. [6] If $m+n+2-g.c.d.(m,n) \leq t \leq m+n+1$, where $g.c.d.(m,n)$ is the greatest common divisor of m and n, then $K_{m,n} \in \mathfrak{T}_t$ for any $m,n \in N$.

Finally, we prove that there are bipartite graphs which have no interval total coloring.

Hertz's graph $H_{k,l}$ ($k \geq 4, l \geq 3$) is defined as follows:

$$V(H_{k,l}) = U \cup V \text{, where}$$

$$U = \{a\} \cup \{c_j^i \mid 1 \leq i \leq k, 1 \leq j \leq l\}, \quad V = \{b_1, b_2, \ldots, b_k, d\},$$

$$E(H_{k,l}) = \{(a,b_i) \mid 1 \leq i \leq k\} \cup \{(b_i, c_j^i) \mid 1 \leq i \leq k, 1 \leq j \leq l\} \cup$$
$$\cup \{(c_j^i, d) \mid 1 \leq i \leq k, 1 \leq j \leq l\}.$$

Clearly, $H_{k,l}$ is a bipartite graph with $\Delta(H_{k,l}) = kl$.

Theorem 10. For any $k \geq 7, l \geq 3$, $H_{k,l} \notin \mathfrak{T}$.

Proof. We show that $H_{k,l}$ has no interval total $t-$coloring, where $t \geq kl+1$. Suppose, to the contrary, that α is an interval total $t-$coloring of $H_{k,l}$. Let $\min S(d,\alpha) = p$, $\alpha\left(\left(c_{j_0}^{i_0}, d\right)\right) = p$ and $\max S(d,\alpha) = q$, $\alpha\left(\left(c_{j_1}^{i_1}, d\right)\right) = q$. Clearly, $q \geq kl + p - 1$. It is easy to see that $\alpha\left(\left(b_{i_0}, c_{j_0}^{i_0}\right)\right) \leq p+2$, thus $\alpha\left(\left(a, b_{i_0}\right)\right) \leq p+l+3$. This implies that

$\alpha\left(\left(a, b_{i_1}\right)\right) \leq p+k+l+3$ and $\alpha\left(\left(b_{i_1}, c_{j_1}^{i_1}\right)\right) \leq p+k+2l+4$,

hence $q = \alpha\left(\left(c_{j_1}^{i_1}, d\right)\right) \leq p+k+2l+6$, which is a contradiction, since

$$kl + p - 1 \leq q = \alpha\left(\left(c_{j_1}^{i_1}, d\right)\right) \leq p+k+2l+6 < kl+p-1,$$

for $k \geq 7, l \geq 3$.

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