# Interval total colorings of complete graphs

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## ABSTRACT

An interval total $t-$coloring of a graph $G$ is a total coloring of $G$ with colors $1,2,\dots,t$ such that at least one vertex or edge of $G$ is colored by color $i$, $i=1,2,\dots,t$, and the edges incident to each vertex $v\in V(G)$ together with $v$ are colored by $d_G(v)+1$ consecutive colors, where $d_G(v)$ is the degree of the vertex $v$ in $G$. In this paper interval total colorings of complete and complete bipartite graphs are investigated.

## Keywords
Total coloring, interval total coloring, complete graph, complete bipartite graph.

## 1. INTRODUCTION

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph $G$, respectively. The degree of a vertex $v\in V(G)$ is denoted by $d_G(v)$, the maximum degree of a vertex of $G$ - by $\Delta(G)$ and the chromatic number of $G$ - by $\chi(G)$. A total coloring of a graph $G$ is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. The total chromatic number $\chi''(G)$ is the smallest number of colors needed for total coloring of $G$. If $\alpha$ is a total coloring of a graph $G$ then $\alpha(v)$ and $\alpha(e)$ denote the color of a vertex $v\in V(G)$ and the color of an edge $e\in E(G)$ in the coloring $\alpha$, respectively. For a total coloring $\alpha$ of a graph $G$ and for any $v\in V(G)$ define the set $S[v,\alpha]$ as follows:

$$S[v,\alpha]\equiv\{\alpha(v)\}\cup\{\alpha(e)|\ e \text{ is incident to } v\}.$$

For two integers $a\le b$ the set $\{a,a+1,\dots,b\}$ is denoted by $[a,b]$.

An interval total $t-$coloring [2,3] of a graph $G$ is a total coloring of $G$ with colors $1,2,\dots,t$ such that at least one vertex or edge of $G$ is colored by color $i$, $i=1,2,\dots,t$, and the edges incident to each vertex $v\in V(G)$ together with $v$ are colored by $d_G(v)+1$ consecutive colors.

For $t\ge 1$ let $\mathfrak{T}_t$ denote the set of graphs which have an interval total $t-$coloring, and assume: $\mathfrak{T}\equiv\bigcup_{t\ge1}\mathfrak{T}_t$. For a graph $G\in\mathfrak{T}$ the least and the greatest values of $t$, for which $G\in\mathfrak{T}_t$, are denoted by $w_\tau(G)$ and $W_\tau(G)$, respectively.

In this paper interval total colorings of complete and complete bipartite graphs are investigated.

The terms and concepts that we do not define can be found in [4,5].

## 2. MAIN RESULTS

**Theorem 1.** For any $n\in N$

1. $K_n\in\mathfrak{T}$,
2. $w_\tau(K_n)=\begin{cases} n, \text{ if } n \text{ is odd,}\\ \frac{3}{2}n, \text{ if } n \text{ is even,}\end{cases}$
3. $W_\tau(K_n)=2n-1$.

**Proof.** Let

$$V(K_n)=\{v_1,v_2,\dots,v_n\} \text{ and } E(K_n)=\{(v_i,v_j)|\ 1\le i<j\le n\}.$$

First of all let us prove that $K_n\in\mathfrak{T}_{2n-1}$ for any $n\in N$. For that we define a total coloring $\alpha$ of the graph $K_n$ in the following way:

1. for $i=1,2,\dots,n$ $\quad\alpha(v_i)=2i-1;$
2. for $i=1,2,\dots,n$ and $j=1,2,\dots,n$, where $i\neq j$,

$$\alpha((v_i,v_j))=i+j-1.$$

It is easy to see that $\alpha$ is an interval total $(2n-1)-$coloring of the graph $K_n$. This proves that $K_n\in\mathfrak{T}$ and $W_\tau(K_n)\ge 2n-1$ for any $n\in N$. On the other hand, it is not difficult to check that $W_\tau(K_n)\le 2n-1$ for any $n\in N$. Hence 1 and 3 hold.

Let us prove 2.

Case 1: $n$ is odd.

Since $K_n$ is a regular graph with $\chi''(K_n)=n$ [1,5] then $w_\tau(K_n)=\chi''(K_n)=n$.

Case 2: $n$ is even.

We now show that $w_\tau(K_n)\le\frac{3}{2}n$.

Define a total coloring $\beta$ of the graph $K_n$ in the following way:

1. for $i = 1, 2, \ldots, \frac{n}{2}$

$$\beta(v_i) = i;$$

2. for $j = \frac{n}{2} + 1, \ldots, n$

$$\beta(v_j) = \frac{n}{2} + j;$$

3. for $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n,\ i < j,\ i + j$ is odd and $i + j - 1 \le n$

$$\beta\left(\left(v_i, v_j\right)\right) = \frac{n}{2} + \frac{i+j-1}{2};$$

4. for $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n,\ i < j,\ i + j$ is odd and $i + j - 1 > n$

$$\beta\left(\left(v_i, v_j\right)\right) = \frac{i+j-1}{2};$$

5. for $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n,\ i < j,\ i + j$ is even and $i + j \le n$

$$\beta\left(\left(v_i, v_j\right)\right) = \frac{i+j}{2};$$

6. for $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n,\ i < j,\ i + j$ is even and $i + j > n$

$$\beta\left(\left(v_i, v_j\right)\right) = \frac{n}{2} + \frac{i+j}{2}.$$

Let us show that $\beta$ is an interval total $\frac{3}{2}n-$coloring of the graph $K_n$.

Let $v_i \in V\left(K_n\right)$, where $i = 1, 2, \ldots, n$.

Case $1'$: $i$ is even.

From 1-6 we have

$$S\left[v_i, \beta\right] = \left(\bigcup_{1\le l\le \frac{n+2-i}{2}} \left\{\frac{n}{2} + \frac{i+(2l-1)-1}{2}\right\}\right) \cup \left(\bigcup_{\frac{n+2-i}{2}<l\le \frac{n}{2}} \left\{\frac{i+(2l-1)-1}{2}\right\}\right) \cup \left(\bigcup_{1\le l\le \frac{n-i}{2},\ l\ne \frac{i}{2}} \left\{\frac{i+2l}{2}\right\}\right) \cup \left(\bigcup_{\frac{n-i}{2}<l\le \frac{n}{2},\ l\ne \frac{i}{2}} \left\{\frac{n}{2} + \frac{i+2l}{2}\right\}\right) \cup \left(i + \frac{n}{2}\cdot sg\left(i - \frac{n}{2}\right)\right) =$$

$$\left[\frac{n+i}{2}, n\right] \cup \left[\frac{n}{2}+1, \frac{n+i}{2}-1\right] \cup \left(\left[\frac{i}{2}+1, \frac{n}{2}\right] \setminus \{i\}\right) \cup \left(\left[n+1, n+\frac{i}{2}\right] \setminus \left\{\frac{n}{2}+i\right\}\right) \cup \left\{i + \frac{n}{2}\cdot sg\left(i - \frac{n}{2}\right)\right\} = \left[\frac{i}{2}+1, n+\frac{i}{2}\right].$$

Case $2'$: $i$ is odd.

From 1-6 we have

$$S\left[v_i, \beta\right] = \left(\bigcup_{1\le l\le \frac{n+1-i}{2}} \left\{\frac{n}{2} + \frac{i+2l-1}{2}\right\}\right) \cup \left(\bigcup_{\frac{n+1-i}{2}<l\le \frac{n}{2}} \left\{\frac{i+2l-1}{2}\right\}\right) \cup \left(\bigcup_{1\le l\le \frac{n+1-i}{2},\ l\ne \frac{i+1}{2}} \left\{\frac{i+2l-1}{2}\right\}\right) \cup \left(\bigcup_{\frac{n+1-i}{2}<l\le \frac{n}{2},\ l\ne \frac{i+1}{2}} \left\{\frac{n}{2} + \frac{i+2l-1}{2}\right\}\right) \cup \left(i + \frac{n}{2}\cdot sg\left(i - \frac{n}{2}\right)\right) =$$

$$\left[\frac{n}{2} + \frac{i+1}{2}, n\right] \cup \left[\frac{n}{2}+1, \frac{i+n-1}{2}\right] \cup \left(\left[\frac{i+1}{2}, \frac{n}{2}\right] \setminus \{i\}\right) \cup \left(\left[n+1, n+\frac{i-1}{2}\right] \setminus \left\{\frac{n}{2}+i\right\}\right) \cup \left\{i + \frac{n}{2}\cdot sg\left(i - \frac{n}{2}\right)\right\} = \left[\frac{i+1}{2}, n+\frac{i-1}{2}\right].$$

This implies that $\beta$ is an interval total $\frac{3}{2}n-$coloring of the graph $K_n$ and, therefore, $w_\tau\left(K_n\right) \le \frac{3}{2}n$.

We next prove that $w_\tau\left(K_n\right) \ge \frac{3}{2}n$.

Suppose, to the contrary, that $\gamma$ is an interval total $w_\tau\left(K_n\right)-$coloring of the graph $K_n$, where $n \le w_\tau\left(K_n\right) \le \frac{3}{2}n - 1$.

Since $w_\tau\left(K_n\right) \ge \chi''\left(K_n\right)$ then $w_\tau\left(K_n\right) \ge n+1$ and, therefore, $n + 1 \le w_\tau\left(K_n\right) \le \frac{3}{2}n - 1$.

Consider the vertices $v_1, v_2, \ldots, v_n$. It is clear that for $i = 1, 2, \ldots, n$

$$1 \le \min S\left[v_i, \gamma\right] \le w_\tau\left(K_n\right) - n + 1.$$

Hence $\left\{w_\tau(K_n)-n+1, \ldots, n\right\} \subseteq S\left[v_i, \gamma\right],\ i = 1, 2, \ldots, n.$

Let us show that none of the vertices $v_1, v_2, \ldots, v_n$ is colored by $j,\ j = w_\tau\left(K_n\right) - n + 1, \ldots, n.$ Suppose that $\gamma\left(v_{i_0}\right) = j_0,\ j_0 \in \left\{w_\tau(K_n)-n+1, \ldots, n\right\}$. It is clear that for $i = 1, 2, \ldots, n,\ i \ne i_0,\ \gamma\left(v_i\right) \ne j_0$. This implies that any vertex $v_i$, except $v_{i_0}$, is incident to an edge of color $j_0$, which is a contradiction. The contradiction shows that $\gamma\left(v_i\right) \notin \left\{w_\tau(K_n)-n+1, \ldots, n\right\}$ for $i = 1, 2, \ldots, n.$ Hence

$$\gamma\left(v_i\right) \in \left\{1, \ldots, w_\tau\left(K_n\right)-n\right\} \cup \left\{n+1, \ldots, w_\tau\left(K_n\right)\right\},\ i = 1, 2, \ldots, n.$$

On the other hand, since $\chi\left(K_n\right) = n$ then

$$\left|\left\{1, \ldots, w_\tau\left(K_n\right)-n\right\}\right| + \left|\left\{n+1, \ldots, w_\tau\left(K_n\right)\right\}\right| \ge n.$$

From this we obtain $w_\tau\left(K_n\right) \ge \frac{3}{2}n$, which is a contradiction.

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**Remark.** A more difficult proof of this theorem was found earlier by P.A. Petrosyan [3].

**Theorem 2.** For any $n \in N$

1. if $2n - 1 \le t \le 4n - 3$ then $K_{2n-1} \in \mathfrak{T}_t$,
2. if $3n \le t \le 4n - 1$ then $K_{2n} \in \mathfrak{T}_t$.

**Proof.** First of all we prove 1. For that we transform an interval total $(4n{-}3)-$coloring $\alpha$ of the graph $K_{2n-1}$, constructed in theorem 1, to interval total $t-$coloring $\beta$ of the same graph.

For every $v \in V(K_{2n-1})$ set:

$$\beta(v)=\begin{cases}\alpha(v), & \text{if } 1\le\alpha(v)\le t,\\ \alpha(v)-2n+1, & \text{if } t+1\le\alpha(v)\le 4n-3.\end{cases}$$

For every $e \in E(K_{2n-1})$ set:

$$\beta(e)=\begin{cases}\alpha(e), & \text{if } 1\le\alpha(e)\le t,\\ \alpha(e)-2n+1, & \text{if } t+1\le\alpha(e)\le 4n-3.\end{cases}$$

It is not difficult to see that $\beta$ is an interval total $t-$coloring of the graph $K_{2n-1}$.

We now prove 2.

Clearly, for the proof 2 it suffices to show that if $3n \le t < 4n-1$ then $K_{2n} \in \mathfrak{T}_t$. For that we transform an interval total $3n-$coloring $\beta$ of the graph $K_{2n}$, constructed in theorem 1, to interval total $t-$coloring $\gamma$ of the same graph.

Define a total coloring $\gamma$ of the graph $K_n$ in the following way:

1. for $i=1,2,\dots,2n$

$$\gamma(v_i)=\begin{cases}\beta(v_i)+(t-3n), & \text{if } \beta(v_i)+(t-3n)\le 2i-1,\\ 2i-1, & \text{if } \beta(v_i)+(t-3n)>2i-1;\end{cases}$$

2. for $i=1,2,\dots,2n-1$, $j=1,2,\dots,2n-1$, $i\ne j$, $i+j-1\le 2(t-3n)+1$

$$\gamma((v_i,v_j)) = i+j-1;$$

3. for $i=1,2,\dots,2n$, $j=1,2,\dots,2n$, $i\ne j$, $2(t-3n)+1 < i+j-1 < 2n$

$$\gamma((v_i,v_j))=\begin{cases}\beta((v_i,v_j))+(t-3n), & \text{if } i+j \text{ is even},\\ \beta((v_i,v_j)), & \text{if } i+j \text{ is odd};\end{cases}$$

4. for $i=1,2,\dots,2n$, $j=1,2,\dots,2n$, $i\ne j$, $2n \le i+j-1 \le 2n+2(t-3n)+1$

$$\gamma((v_i,v_j)) = i+j-1;$$

5. for $i=3,4,\dots,2n$, $j=3,4,\dots,2n$, $i\ne j$, $i+j-1 > 2n+2(t-3n)+1$

$$\gamma((v_i,v_j))=\begin{cases}\beta((v_i,v_j))+(t-3n), & \text{if } i+j \text{ is even},\\ \beta((v_i,v_j)), & \text{if } i+j \text{ is odd}.\end{cases}$$

It is not difficult to see that $\gamma$ is an interval total $t-$coloring of the graph $K_{2n}$.

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In [3] P.A. Petrosyan proved the following:

**Theorem 3.** If $m+n+2-g.c.d.(m,n)\le t\le m+n+1$, where $g.c.d.(m,n)$ is the greatest common divisor of $m$ and $n$, then $K_{m,n}\in\mathfrak{T}_t$ for any $m,n\in N$.

**Theorem 4.** For any $m,n\in N$

$$W_\tau(K_{m,n})=\begin{cases}m+n+1, & \text{if } m=n=1,\\ m+n+2, & \text{otherwise.}\end{cases}$$

**Proof.** Let $V(K_{m,n}) = U\cup V$, where $U=\{u_1,u_2,\dots,u_m\}$, $V=\{v_1,v_2,\dots,v_n\}$ and $E(K_{m,n})=\{(u_i,v_j)\,|\,1\le i\le m, 1\le j\le n\}$.

Clearly, theorem is true for the case $m=n=1$.

Assume that $m\ge n$ and $m\ne 1$.

We first show that $W_\tau(K_{m,n})\ge m+n+2$.

Define a total coloring $\lambda$ of the graph $K_{m,n}$ in the following way:

1. $\lambda(u_1)=1$ and for $i=2,3,\dots,m$

$$\lambda(u_i)=n+2+i;$$

2. for $j=1,2,\dots,n$

$$\lambda(v_j)=j+2;$$

3. for $k=1,2,\dots,n$

$$\lambda((u_1,v_k))=k+1;$$

4. for $i=2,3,\dots,m$ and $j=1,2,\dots,n$

$$\lambda((u_i,v_j))=i+j+1.$$

It is easy to see that $\lambda$ is an interval total $(m+n+2)-$coloring of the graph $K_{m,n}$.

We next show that $W_\tau(K_{m,n})\le m+n+2$.

Suppose, to the contrary, that $\mu$ is an interval total $W_\tau(K_{m,n})-$coloring of the graph $K_{m,n}$, where $W_\tau(K_{m,n})\ge m+n+3$. We distinguish the following four possible cases:

1. there are edges $e,e'\in E(K_{m,n})$ such that $\mu(e)=1$, $\mu(e')=W_\tau(K_{m,n})$;
2. there is a vertex $w\in V(K_{m,n})$ and there is an edge $e\in E(K_{m,n})$ such that $\mu(w)=1$, $\mu(e)=W_\tau(K_{m,n})$;
3. there is an edge $e\in E(K_{m,n})$ and there is a vertex $w\in V(K_{m,n})$ such that $\mu(e)=1$, $\mu(w)=W_\tau(K_{m,n})$;
4. there are vertices $w,w'\in V(K_{m,n})$ such that $\mu(w)=1$, $\mu(w')=W_\tau(K_{m,n})$.

Case 1: $\mu(e)=1$, $\mu(e')=W_\tau(K_{m,n})$.

Let $e=(u_1,v_1)$, $e'=(u_2,v_2)$, where $u_1,u_2\in U$, $v_1,v_2\in V$. Clearly, $u_1\ne u_2$ and $v_1\ne v_2$. Note that $\mu((u_1,v_2))\le n+1$. This implies that $W_\tau(K_{m,n})= =\mu((u_2,v_2))\le m+n+1$, which is a contradiction.

Case 2: $\mu(w)=1$, $\mu(e)=W_\tau(K_{m,n})$.

Let $e=(u,v)$. If $w\in U$ then $\mu((w,v))\le n+1$ and, thus $W_\tau(K_{m,n})=\mu((u,v))\le m+n+1$, which is a contradiction. If $w\in V$ then $\mu((u,w))\le m+1$ and, thus $W_\tau(K_{m,n})=\mu((u,v))\le m+n+1$, which is a contradiction.

Case 3: $\mu(e)=1,\ \mu(w)=W_\tau\left(K_{m,n}\right)$.

It is easy to see that a total coloring $\varphi$, where

1. $\forall w\in V(K_{m,n})\quad \varphi(w)=W_\tau\left(K_{m,n}\right)+1-\mu(w)$,
2. $\forall e\in E(K_{m,n})\quad \varphi(e)=W_\tau\left(K_{m,n}\right)+1-\mu(e)$,

is also an interval total $W_\tau\left(K_{m,n}\right)$ – coloring of the graph $K_{m,n}$. For the proof of the case it suffices to note that a coloring $\varphi$ meets the case 2.

Case 4: $\mu(w)=1,\ \mu(w')=W_\tau\left(K_{m,n}\right)$.

If $w\in U,\ w'\in V$ or $w\in V,\ w'\in U$ then $\mu\left((w,w')\right)\le n+1$ ( $w\in U,\ w'\in V$ ) or $\mu\left((w',w)\right)\le m+1$ ( $w\in V,\ w'\in U$ ) and, thus $W_\tau\left(K_{m,n}\right)=\mu\left(w'\right)\le m+n+1$, which is a contradiction.

If $w,w'\in U$ then $\mu\left((w,v_i)\right)\le n+1,\ i=1,\ldots,n$, thus $\mu\left((w',v_i)\right)\le m+n+1,\ i=1,\ldots,n,$ and $W_\tau\left(K_{m,n}\right)=$ $=\mu\left(w'\right)\le m+n+2$, which is a contradiction.

If $w,w'\in V$ then $\mu\left((u_j,w)\right)\le m+1,\ j=1,\ldots,m$, thus $\mu\left((u_j,w')\right)\le m+n+1,\ j=1,\ldots,m,$ and $W_\tau\left(K_{m,n}\right)=$ $=\mu\left(w'\right)\le m+n+2$, which is a contradiction.

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Theorem 3 and theorem 4 imply

**Corollary.** For any $n\in N$

1. $K_{n,n}\in\mathfrak{T}$,
2. $w_\tau\left(K_{n,n}\right)=n+2$,
3. $W_\tau\left(K_{n,n}\right)=\begin{cases}2n+1, & \text{if } n=1,\\ 2n+2, & \text{if } n\ge 2,\end{cases}$
4. if $w_\tau\left(K_{n,n}\right)\le t\le W_\tau\left(K_{n,n}\right)$ then $K_{n,n}\in\mathfrak{T}_t$.

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