# Generating New Boundary Elements of Numerical Characterization of n-cube Subset Partitioning

Hasmik Sahakyan
Institute For Informatics and Automation Problems
Yerevan, Armenia
e-mail: hasmik@ipia.sci.am

Levon Aslanyan
Information Society Technologies Center
Yerevan, Armenia
e-mail: lasl@sci.am

## ABSTRACT
Generating numerical characteristics of subsets partitions of $n$ dimensional unit cube $E^n$ is considered as a recursion of constructing the same characteristics in $n-1$ dimensional unit cube. The research focuses on a particular case leading to more boundary elements of numerical characterization which possess also an additional property.

## Keywords
Monotone Boolean functions, Set Systems, (0,1)-matrices

## 1. INTRODUCTION
For a given vertex subset of $n$-dimensional unite cube $E^n$, its numerical characterization is composed by partitions and their sizes, which further serve as coordinates of the corresponding associated vector of partitions. General numerical characterization of vertex subsets of $E^n$ through their partitions is considered in [8], where a complete and simple structural description of the set of all associated vectors of partitions is given. The description is given by means of the set of its boundary elements. Elements of the boundary sets with minimum and maximum weight are known by [5,6]. Current research focuses on generating more boundary elements via the known ones for smaller dimensions, and composing corresponding subsets of vertices.

## 2. BASIC STRUCTURAL DESCRIPTION
Let $E^n$ be the set of vertices of $n$-dimensional unit cube $E^n=\{(x_1,\cdots,x_n)/x_i\in\{0,1\}, i=1,\cdots,n\}$. For an arbitrary variable $x_i$, consider partition of the cube into two subcubes according to the value of $x_i, 1\le i\le n$. Denote these subcubes by $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$ correspondingly. Similarly, each subset of vertices $M\subseteq E^n$ will be partitioned into $M_{x_1=1}$ and $M_{x_1=0}$. For a given $m$, $0\le m\le 2^n$, consider an $m$-vertex subset $M$ of $E^n$. The vector $S=(s_1,\cdots,s_n)$ is called associated vector of partitions for $M$ if $s_i=|M_{x_1=1}|$ for all $1\le i\le n$. Let $\psi_m(n)$ denote the set of all associated vectors of partitions of $m$-subsets of $E^n$. Let $\Xi^n_{m+1}$ denote the $n$ dimensional $(m+1)$-valued grid, i.e., the set of all integer-valued vectors $S=(s_1,s_2,\cdots,s_n)$ with $0\le s_i\le m$, $i=1,\cdots,n$. Obviously $\psi_m(n)\subseteq\Xi^n_{m+1}$.

A vector $S\in\psi_m(n)$ is called an upper (lower) boundary vector for $\psi_m(n)$ if no vector $R\in\Xi^n_{m+1}$ with $R>S$ ($S>R$) belongs to $\psi_m(n)$. By $\hat{\psi}_m(n)$ and $\breve{\psi}_m(n)$, respectively, denote the sets of all upper and lower boundary vectors of $\psi_m(n)$. The sets $\hat{\psi}_m(n)$ and $\breve{\psi}_m(n)$ contain equal numbers of elements [8]. Let $r$ denote the number of elements of each of these two boundary sets and let $\hat{\psi}_m(n)=\{\hat{S}^1,\cdots,\hat{S}^r\}$ and $\breve{\psi}_m(n)=\{\breve{S}^1,\cdots,\breve{S}^r\}$, such that $(\hat{S}^j,\breve{S}^j)$ is a pair of complementary vectors: $\hat{s}^j_i=m-\breve{s}^j_i$, for $1\le i\le n$. All coordinates of each vector $\hat{S}^j$ are greater than or equal to $m/2$ and all coordinates of each vector $\breve{S}_j$ are less than or equal to $m/2$. For a pair of complementary vectors ($\hat{S}^j,\breve{S}^j$), let $I(\hat{S}_j)$ (the notion $I(\breve{S}_j)$ also may be used) denote the vectors of the sub-cube in $\Xi^n_{m+1}$ spanned by these vectors: $I(\hat{S}_j)=\{Q\in\Xi^n_{m+1}/\breve{S}_j\le Q\le\hat{S}_j\}$. The following theorem proves that the collection of all sub-cubes $\{I(\hat{S}_j)/\hat{S}_j\in\hat{\psi}_m(n)\}$ composes $\psi_m(n)$:

**Theorem 1.** $\psi_m(n)=\bigcup_{j=1}^{r} I(\hat{S}_j)$

## 3. BOUNDARY CASES
The basic object of study in this section is the set of all monotone Boolean functions (in terms of $E^n$) with exactly $m$ ones. Let $M^1_m(n)$ be the set of all associated vectors of partitions of $m$-subsets corresponding to the ones of these monotone Boolean functions. Similarly $M^0_m(n)$ is the set of all associated vectors of partitions of $m$-subsets corresponding to the zeros of the monotone Boolean functions. It is easy to check that $\hat{\psi}_m(n)\subseteq M^1_m(n)$ and $\breve{\psi}_m(n)\subseteq M^0_m(n)$. Consequently for the description of

$\psi_m(n)$ it is sufficient to find those monotone Boolean functions which correspond to $\widehat{\psi}_m(n)$ (and/or $\breve{\psi}_m(n)$).

Suppose the weights (sum of all coordinates) of the vectors of $\widehat{\psi}_m(n)$ belong to some interval $[L_{\min}, L_{\max}]$. A specific set of monotone Boolean functions is constructed in [6], for which the corresponding associated vectors of partitions belong to $\widehat{\psi}_m(n)$ and have weight equal to $L_{\min}$. The value of $L_{\min}$ and the coordinates of associated vectors are also analyzed in detail.

Let $D^{i_1,\cdots,i_n}$ be the set of vertices of $E^n$ arranged in decreasing order of numeric value of the binary vector $< x_{i_1}, \cdots, x_{i_n} > \in E^n$. It is easy to check that the first $2^{n-1}$ elements of $D^{i_1,\cdots,i_n}$ form the set of vertices of $E^{n-1}_{x_{i_1}=1}$ and the reminder $2^{n-1}$ elements form $E^{n-1}_{x_{i_1}=0}$, being arranged in the same decreasing order of numeric value of $< x_{i_2}, \cdots, x_{i_n} >$. Denote these sets by $D^{i_2\cdots,i_n}_{x_{i_1}=1}$ and $D^{i_2\cdots,i_n}_{x_{i_1}=0}$ respectively. Similarly the first $2^{n-2}$ elements of $D^{i_2\cdots,i_n}_{x_{i_1}=1}$ form $E^{n-2}_{x_{i_1}=1, x_{i_2}=1}$ and the reminder $2^{n-2}$ elements form $E^{n-2}_{x_{i_1}=1, x_{i_2}=0}$ - arranged in decreasing order of numeric value of the vector $< x_{i_3}, \cdots, x_{i_n} >$, etc. It follows that initial parts (subsets) of $D^{i_1,\cdots,i_n}$ (similarly of $D^{i_2\cdots,i_n}_{x_{i_1}=1}$, $D^{i_2\cdots,i_n}_{x_{i_1}=0}$, etc.) of arbitrary sizes serve as sets of ones of some monotone Boolean functions of proper dimension. Let $D^{i_1,\cdots,i_n}(m)$ denote the $m$-length initial part of $D^{i_1,\cdots,i_n}$. Denote by $D^n$ the overall set of all enumerations $D^{i_1,\cdots,i_n}$, and let $D^n(m)$ be the set of corresponding $m$-length initial parts.

Denote the class of monotone Boolean functions, with sets of ones belonging to $D^n(m)$ by $f^{D^n(m)}$ and let $S^{D^n(m)}$ be the corresponding set of associated vectors of partitions.

**Note.**

Let us consider a vector $S = (s_1, \cdots, s_n)$ of $S^{D^n(m)}$, and let it be the associated vector of partitions for some $D^{i_1,\cdots,i_n}(m)$. It is easy to check that $s_1 \geq s_2 \geq \cdots \geq s_n$. Further, vector $S$ obeys the following very useful property: any vector $(s_1, \cdots, s_j + 1, *, \cdots, *)$ for an arbitrary $j, 1 \leq j \leq n$, does not belong to $\psi_m(n)$ (* means an arbitrary value of the coordinate).

**Theorem 2**

The weight of a vector $S$ of $\widehat{\psi}_m(n)$ is equal to $L_{\min}$ if and only if $S \in S^{D^n(m)}$.

The proof of theorem which is not the target to bring here is by induction on $n$.

## 3. NEW BOUNDARY ELEMENTS

This point considers the problem of finding new monotone Boolean functions, which correspond to the elements of $\widehat{\psi}_m(n)$. Consider the way of composing the target constructions for $n$ through same type structures which have been constructed in dimension $n-1$: consider partition of $E^n$ according to some variable, and examine pairs of monotone Boolean functions (one in each subcube), which correspond to the upper boundary vectors - to discover such pairs, that in $E^n$ (their union in $E^n$) may correspond to $\widehat{\psi}_m(n)$.

For a given $m$ consider its arbitrary partition: $m = m_1 + m_2$, with the only requirement that $2^{n-1} \geq m_1 \geq m_2 \geq 0$. Partition the $E^n$ according to the value of $x_i$, and consider arbitrary monotone Boolean functions in $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$ with $m_1$ and $m_2$ ones respectively. We intend to find feasible pairs of monotone functions to get at first monotone functions in $E^n$, and then - monotone functions which correspond to $\widehat{\psi}_m(n)$.

We restrict ourselves with the following particular case.

In $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$ consider monotone functions for which associated vectors belong to sets $\widehat{\psi}_{m_1}(n-1)$ and $\widehat{\psi}_{m_2}(n-1)$ respectively, and have the minimum possible weights. By theorem 2 these are functions of the form $f^{D^{n-1}(m_1)}$ and $f^{D^{n-1}(m_2)}$ respectively. Consider the case of the same order of variables, let it be $D^{2,\cdots,n}(m_1)$ and $D^{2,\cdots,n}(m_2)$. Denote by $S' = (s'_2, \cdots, s'_n)$ and $S'' = (s''_2, \cdots, s''_n)$ - the corresponding associated vectors. Obviously this pair is feasible for getting a monotone function in $E^n$. It turned out that it is feasible for getting an upper boundary vector as well, that is vector $S = (m_1, s'_2 + s''_2, \cdots, s'_n + s''_n)$, which belongs to $\widehat{\psi}_m(n)$.

The choice of this particular case is caused by theorem 2 which provides the description of monotone Boolean functions corresponding to the upper boundary vectors of minimum weight.

**Theorem 3**

$S = (m_1, s'_2 + s''_2, \cdots, s'_n + s''_n) \in \widehat{\psi}_m(n)$.

The theorem can be proven by contradictory assumption considering possible cases.

Vector $S = (m_1, s'_2 + s''_2, \cdots, s'_n + s''_n)$ constructed above, has another important property, - it is a **steepest** vector. This is an important property because as proven in [7] for the description of $\psi_m(n)$ it is sufficient to find steepest boundary vectors.

**Steepest Vectors** [3]:

Let $S=(s_1,\cdots,s_n)$ and $Q=(q_1,\cdots,q_n)$ be two vectors of length $n$ with integer, nonnegative components, and let $s_1\geq\cdots\geq s_n$ and $q_1\geq\cdots\geq q_n$. $Q$ is an elementary flattening of $S$ if and only if $Q$ can be obtained from $S$ by:

(1) finding $i,j$ such that $s_i\geq s_j+2$ ; and then

(2) transferring 1 from the larger to the smaller; that is, taking $q_i=s_i-1$ and $q_j=s_j+1$ ; and then

(3) re-ordering the resulting sequence so that it is decreasing.

Observe that operation of elementary flattening doesn't change weights of vectors.

We say that $Q$ is flatter than $S$, and $S$ is steeper than $Q$, if and only if $Q$ can be obtained from $S$ by a non-empty sequence of elementary flattening. $S$ is a steepest vector if and only if there is no vector in $\psi_m(n)$, which is steeper than $S$.

It follows from these definitions, that the vectors of $S^{D^n(m)}$ are steepest vectors.

Coming back to the statement of Theorem 3 we formulate:

**Theorem 4**

Vector $S=(m_1,s'_2+s''_2,\cdots,s'_n+s''_n)=(s_1,s_2,\cdots,s_n)$ is a steepest vector.

The proof of this property is based on the fact that vectors $S'=(s'_2,\cdots,s'_n)$ and $S''=(s''_2,\cdots,s''_n)$, being from $D^{2,\cdots n}(m_1)$ and $D^{2,\cdots n}(m_2)$ respectively, are steepest vectors and it follows from the note given above that vector $S$ is steepest by means of the coordinates $s_2,\cdots,s_n$. Unfeasibility to make $S$ steeper through the pairs $(s_1,s_i)$ can be proven considering partition of $E^n$ according to the values of $x_1$ and $x_i$ simultaneously.

Finally we bring the constructive procedure to compose the $m$-subsets of vertices which have the prescribed associated vector of partitions $S=(m_1,s'_2+s''_2,\cdots,s'_n+s''_n)$.

Known that the sets $D^{2,\cdots n}(m_1)$ and $D^{2,\cdots n}(m_2)$ can be constructed by the "interval bisection" method [6], then constructing the required set can be provided in the following way: first construct a column, putting $m_1$ consecutive ones and ($m-m_1$) consecutive zeros; continue the construction in this 2 sections by the same "interval bisection" method constructing the column 2, etc.

The weight of $S$ and the values of coordinates can be calculated also.

# 4. CONCLUSION

Generating numerical characteristics of subsets partitions of $n$ dimensional unit cube $E^n$ is considered as a recursion of constructing the same characteristics in $n-1$ dimensional unit cube, for a particular case, which leads to more boundary elements.

# REFERENCES


[1] H. J. Ryser, "Combinatorial Mathematics", The Mathematical Association of America, p. 150, 1963.

[2] L. Aslanyan, "Isoperimetry problem and related extremal problems of discrete spaces", *Problemy Kibernetiki, 36*, pp. 85-126, 1976.

[3] Billington D., "Conditions for degree sequences to be realisable by 3-uniform hypergraphs". *The Journal of Combinatorial Mathematics and Combinatorial Computing"*. 3, 1988, pp. 71-91.

[4] R.J.Gardner, P. Gritzmann, D. Prangenberg, "On the computational complexity of reconstructing lattice sets from the X-rays", *Technical Report (970-05012)*, Techn.Univ. Munchen, 1997.

[5] L. Aslanyan, H. Sahakyan, "On the boundary cases of partitioning of subsets of the n-dimensional unit cube", *Computer Science & Information Technologies Conference*, Yerevan, September 17-20, pp. 164-166, 2001.

[6] H. Sahakyan, L. Aslanyan, "The boundary descriptors of the n-dimensional unit cube subset partitioning", *Journal "Information Theories and Applications", ISSN 1310-0513*, Volume 13 (2006), Number 3.

[7] H. Sahakyan, "Description Reduction for restricted sets of (0,1) matrices" "Information Research, Applications and Education", *i.Tech Proceedings, ISBN:954-16-0036-0,* FOI Commerce, Sofia, 2006

[8] H. Sahakyan, "Numeral characterization of n-cube subset partitioning", *Discrete Applied Mathematics*, vol.157 (2009), issue 9, pp. 2191-2197.