The Number of Solutions of Some Type of System of Inequalities With Boolean Variables

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Abstract

The problem of finding of the number of Boolean solutions of system of inequalities of some type is studied. Using the theorem on necessary and sufficient condition for the existence of solution of similar Boolean systems of equations, it is proved that the above problem can be reduced to the task of finding the number of solutions satisfying certain conditions for simple system of inequalities. It is proved also that the latter problem in its turn can be reduced to the problem of finding of the number of paths in rectangular grid that satisfy some limitations.

For the correspondingly defined metric in the binary symmetric additive noisy communication channel for certain types of error vectors [1], [2], [3] it is considered the problem of the finding of the power of ball or spherical surface according to those error vectors. In [2], [3] are studied such $\overline{e} = (e_1, e_2, \dots, e_n)$ error vectors for which the following system of inequalities takes place:

$$\begin{cases} e_1 + e_2 + \dots + e_k \le a \\ e_2 + e_3 + \dots + e_{k+1} \le a \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ e_{n-k+1} + e_{n-k+2} + \dots + e_n \le a \end{cases}, \qquad (1)$$

where $a \le k \le n$, $e_i \in \{0,1\}$, $i = 1,2,\dots,n$. The number of Boolean solutions of system [1] is denoted by $\phi(n,k,a)$. Obviously, in the case of $k = n$ such error vectors coincide with error vectors correcting *a* errors. In the case of $k < n$ in [1] the problem of finding of the number of Boolean solutions of (1) arises while estimating the code power with the method of Hemming spherical package. a problem occurs to find the number of Boolean solutions to this system. In [4] it is proved that the problem of finding of the number of Boolean solutions to this system in the case of $k \ge \frac{n}{2}$, can be reduced to the case when $n = 2p$, $k = p$.

On the other hand, the problem of finding of the number of solutions for system (1) can be reduced to the similar problem for the system of inequalities obtained from system (1) by replacing $\le$ by $\ge$.

$$\begin{cases} e_1 + e_2 + \dots + e_k \ge a \\ e_2 + e_3 + \dots + e_{k+1} \ge a \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ e_{n-k+1} + e_{n-k+2} + \dots + e_n \ge a \end{cases} \qquad (2)$$

We denote by $\theta(n,k,a)$ the number of Boolean solutions to system (2). Denote by $\varphi(n,a)$ the number of Boolean solutions to the following system

$$\begin{cases} x_1 + x_2 + \dots + x_n \ge a \\ x_2 + x_3 + \dots + x_{n+1} \ge a \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_{n+1} + x_{n+2} + \dots + x_{2n} \ge a \end{cases}.$$

As it was shown in [4] the following relation hold

$$\phi(n,k,a) = \sum_{l=0}^{2k-n} \binom{2k-n}{l} \varphi(n-k, k-a-l). \qquad (3)$$

Using the theorem on necessary and sufficient condition for the existence of solution of similar Boolean systems of equations we get the following

Theorem 1. Let $\Psi(i,j,p,n,a)$ be the number of vectors $\overline{a} = (a_1, a_2, \dots, a_{n+1})$ beginning with $a_1 = p$ and satisfied with the following conditions

$|a_k - a_{k+1}| \le 1$, $k = 1,2,\dots,n$,

$|\{k / a_k - a_{k+1} = 1, \quad k = 1,2,\dots,n\}| = i$,

$|\{k / a_k - a_{k+1} = -1, \quad k = 1,2,\dots,n\}| = j$,

$a \le a_k \le n$, $k = 1,2,\dots,n$.

Then we have

$$\varphi(n,a)=\sum_{p=0}^{n}\sum_{i=0}^{p}\sum_{j=0}^{n-p}\binom{n-i-j}{p-i}\Psi(i,j,p,n,a). \qquad (4)$$

Now let us consider the following system of inequalities

$$\begin{cases} x_1 \le a \\ x_1+x_2 \le a \\ \text{.........................} \\ x_1+x_2+\ldots+x_n \le a \end{cases}, \qquad (5)$$

where $x_k \in \{-1,0,1\}$, $k=1,2,\ldots,n$.

In the grid of size $k \times n$ $(k \ge n)$ the points $O(0,0)$, $A(k,n)$ and $B(n,n)$ are marked (Figure 1).The number of paths joining the points $O(0,0)$ and $A(k,n)$ which lie below the segment OB is required to find. This number we denote by $V(k,n)$.

Lemma 1. We have that

$$V(k,n)=\binom{k+n}{n}-$$
$$-\sum_{i=0}^{n-1}\left(\frac{1}{i+1}\binom{2i}{i}\binom{k+n-2i-1}{k-i}\right).$$

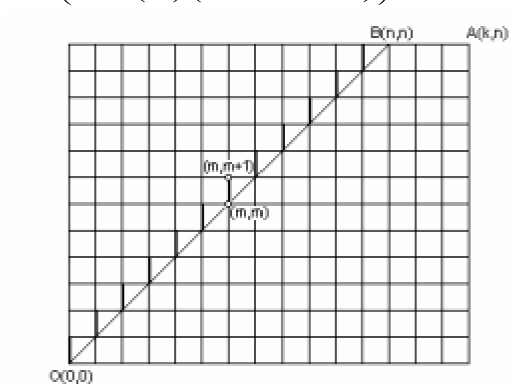

Figure 1

In the grid of size $k \times n$ $(k \ge n-a)$ the points $O(0,0)$, $A(k,n)$, $B(n-a,n)$ and $C(0,a)$ are marked (Figure 2).

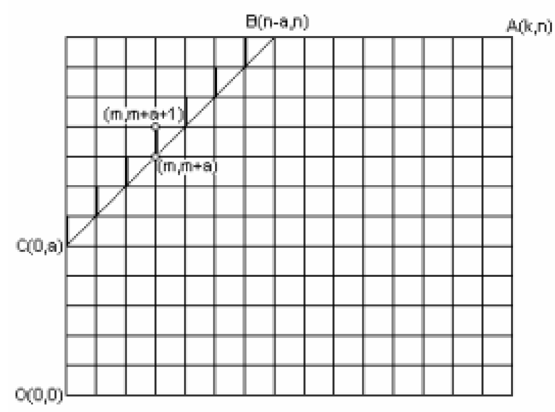

Figure 2

The number of paths joining the points $O(0,0)$ and $A(k,n)$ is required to find which lie below the segment CB. This number we denote by $W(a,k,n)$.

Lemma 2. We have that

$$W(a,k,n)=\binom{k+n}{n}-$$
$$-\sum_{j=0}^{n-a-1}\binom{k+n-2j-a-1}{k-j}\left(\binom{2j+a}{j}-\right.$$
$$\left.-\sum_{i=0}^{j-1}\frac{1}{i+1}\binom{2i}{i}\binom{2j+a-2i-1}{j+a-i}\right).$$

We denote the number of solutions to system (5) with i number of 1 and j number of -1 of x_k's by $Q(i,j,n,a)$.

Theorem 2. We have that

$$Q(i,j,n,a)=\begin{cases}\binom{i+j}{i}\binom{n}{n-i-j}, if\ i \le a, \\ W(a,j,i)\binom{n}{n-i-j}, if\ i > a. \end{cases} \qquad (6)$$

It is obvious that the following equality holds

$$\Psi(i,j,p,n,a)=Q(i,j,n,p-a). \qquad (7)$$

We get the value $\varphi(n,a)$ from the equalities (4), (6) and (7). Then putting this value of $\varphi(n,a)$ into expression (3) we get finally

$$\phi(n,k,a) = \sum_{l=0}^{2k-n} \binom{2k-n}{l} \varphi(n-k, k-a-l) =$$

$$= \sum_{l=0}^{2k-n} \binom{2k-n}{l} \times$$

$$\times \sum_{p=k-a-l}^{n-k} \left\{ \sum_{v=0}^{p-k+a+l} \sum_{u=0}^{n-k-p} \left(\binom{n-k-u-v}{p-v} \times \right. \right.$$

$$\left. \times \binom{u+v}{v} \binom{n-k}{n-k-u-v} \right) +$$

$$+ \sum_{v=p-k+a+l+1}^{p} \sum_{u=v-p+k-a-l}^{n-k-p} \left[\binom{n-k-u-v}{p-v} \times \right.$$

$$\times \binom{n-k}{n-k-u-v} \left[\binom{u+v}{v} - \right.$$

$$- \sum_{j=0}^{v-p+k-a-l-1} \left\{ \binom{u+v-2j-p+k-a-l-1}{u-j} \right.$$

$$\times \left[\binom{2j+p-k+a+l}{j} - \right.$$

$$- \sum_{i=0}^{j-1} \frac{1}{i+1} \binom{2i}{i} \times$$

$$\times \binom{2j+p-k+a+l-2i-1}{j+p-k+a+l-i} \Bigg) \Bigg) \Bigg\} \Bigg) \Bigg) \Bigg\} \Bigg\}$$

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