Hypothesis Testing for Arbitrarily Varying Source with Multiterminal Data Compression

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ABSTRACT
The multiterminal hypothesis testing for arbitrarily varying sources (AVS) is considered. This is an extension of Han's [1] and Ahlswede-Csiszár [2] schemes for a more general class of sources. In part, the solutions can be easily specialized for earlier known particular cases.

Keywords
Hypothesis testing, data compression, arbitrarily varying source.

1. INTRODUCTION
The history of hypothesis testing (HT) with communication constraints goes back to Ahlswede and Csiszár [2]. The problem of HT for AVS's was solved by Fu and Shen [3]. In that paper they generalized Stein's lemma, and later investigated exponential-type constraints in HT for that source model in [4]. The latter study had implications for AVS coding problem, with further advancement in [5]. AVS's are studied also in [6] in terms of logarithmically asymptotically optimal testing.

HT problem with multiterminal data compression was considered by Han [1] as an extension of Ahlswede and Csiszár [2] setting.

In our turn, instead of correlated discrete memoryless source (DMS) studied in [1] we treat the arbitrarily varying correlated one to generalize the solutions for HT under the multiterminal data compression.

2. DEFINITIONS
Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{S}$ be finite sets. $\mathcal{X}$ and $\mathcal{Y}$ stand for alphabets of sources, which are correlated with a law defined by correlation state $s \in \mathcal{S}$, the latter being the alphabet for states. Let $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\mathcal{Y})$ be the set of probability distributions (PDs) on $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{P}(\mathcal{X}\times\mathcal{Y})$ be the set of joint PDs on $\mathcal{X}\times\mathcal{Y}$.

An arbitrarily varying correlated source (AVCS) $\{X, Y, \mathcal{S}\}$ is defined by the family of distributions on $\mathcal{P}(\mathcal{X}\times\mathcal{Y})$ subject to correlation states $s \in \mathcal{S}$:

$$\mathcal{W}_{XY} \stackrel{\triangle}{=} \{W_{s,XY}, \quad s \in \mathcal{S}\} \subset \mathcal{P}(\mathcal{X}\times\mathcal{Y}) \qquad (1)$$

where

$$W_{s,XY} \stackrel{\triangle}{=} \{W_{XY}(x,y|s), \quad x,y \in \mathcal{X}\}.$$

The corresponding marginal ones are

$$W_{s,X} \stackrel{\triangle}{=} \{W_X(x|s),\ x \in \mathcal{X}\}$$

$$W_{s,Y} \stackrel{\triangle}{=} \{W_Y(y|s),\ y \in \mathcal{Y}\}.$$

They are grouped into the families

$$\mathcal{W}_X \stackrel{\triangle}{=} \{W_{s,X}, \quad s \in \mathcal{S}\},$$

$$\mathcal{W}_Y \stackrel{\triangle}{=} \{W_{s,Y}, \quad s \in \mathcal{S}\}.$$

Logically, $\mathcal{W}_X$ and $\mathcal{W}_Y$ make AVS's.

The joint probability $W^N_{\mathbf{s},\mathbf{XY}}$ of pair $(\mathbf{x},\mathbf{y})$, $\mathbf{x} \stackrel{\triangle}{=} (x_1,\cdots,x_N) \in \mathcal{X}^N$, $\mathbf{y} \stackrel{\triangle}{=} (y_1,\cdots,y_N) \in \mathcal{Y}^N$, emitted from the AVCS stayed at sequential states $\mathbf{s} \stackrel{\triangle}{=} (s_1,\cdots,s_N) \in \mathcal{S}^N$ is measured by

$$W^N_{\mathbf{s},\mathbf{XY}}(\mathbf{x},\mathbf{y}) \stackrel{\triangle}{=} \prod_{n=1}^{N} W_{XY}(x_n,y_n|s_n), \quad \mathbf{s} \in \mathcal{S}^N. \qquad (2)$$

The code consists of the encodings $f_1 : \mathcal{X}^N \to \mathcal{L}^1_N(\subseteq \mathcal{X}^N)$, $f_2 : \mathcal{Y}^N \to \mathcal{L}^2_N(\subseteq \mathcal{Y}^N)$ and decoding $g_1 : \mathcal{L}^1_N \to \mathcal{X}^N$, $g_2 : \mathcal{L}^2_N \to \mathcal{Y}^N$ mappings, where $|\mathcal{L}^l_N|(||f_l||)$, $l = 1,2$, are the volumes of the codes.

Suppose that the statistician observes AVCS indirectly, via encoding functions of rate R_l, $l = 1,2$, that is instead of sample $\mathbf{x}$ only $f_1(\mathbf{x})$ is available and instead of sample $\mathbf{y}$ is $f_2(\mathbf{y})$ available, where

$$\frac{1}{N}\log ||f_l|| \le R_l.$$

Testifying the compressed data within the binary HT scheme with communication constraints for unknown AVCS $\mathcal{W}$, in favor of one of two hypotheses

$$H_0 : \quad \mathcal{W} = \mathcal{W}_{XY} \qquad (3)$$

$$H_1 : \quad \mathcal{W} = \mathcal{G}_{XY} \qquad (4)$$

should be made a decision, where $\mathcal{G}_{XY}$ is another set of distributions family defined similarly to (1).

A test φ^N by the statistician is a partition of $\mathcal{X}^N$ into two disjoint subsets $\mathcal{A}$ and $\mathcal{A}^c$. If $\mathbf{x} \in \mathcal{A}$ then the test adopts the hypothesis H_0, otherwise the alternative H_1. So the test φ^N and the sets $(\mathcal{A},\mathcal{A}^c)$ are equivalent concepts.

In decision making in favor of one of two alternatives the following errors may occur: the hypothesis H_l is adopted, but the correct is H_k, $l \ne k$, $l = 1,2$. We consider the maximum probability of such errors over all state sequences from $\mathcal{S}^N$. Then, the first kind error probability is

$$\alpha_{\mathrm{AVS}}(\varphi^N, \mathcal{W}_{XY}) \stackrel{\triangle}{=} \max_{\mathbf{s}\in\mathcal{S}^N} W^N_{\mathbf{s},f_1(\mathbf{X})f_2(\mathbf{Y})}(\mathcal{A}^c),$$

and the error probability of the second kind is

$$\overline{\beta}_{\mathrm{AVS}}(\varphi^N,\mathcal{G}_{XY}) \triangleq \max_{\mathbf{s}\in\mathcal{S}^N} G^N_{\mathbf{s},f_1(\mathbf{X})f_2(\mathbf{Y})}(\mathcal{A}). \tag{5}$$

Let

$$\beta_{\mathrm{AVS}}(N,\varepsilon,f_1,f_2,\mathcal{W}_{XY},\mathcal{G}_{XY}) \triangleq \min_{\mathcal{A}\subset\mathcal{L}^1_N\times\mathcal{L}^2_N}\{\overline{\beta}_{\mathrm{AVS}}(\varphi^N,\mathcal{G}_{XY}) \\ \alpha_{\mathrm{AVS}}(\varphi^N,\mathcal{W}_{XY})\geq 1-\varepsilon,\ \forall\mathbf{s}\in\mathcal{S}^N\}, \tag{6}$$

$$\beta_{\mathrm{AVS}}(R_1,R_2,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY}) \triangleq \min_{f_l:\,\log||f_l||\leq NR_l}\{\beta_{\mathrm{AVS}}(N,\varepsilon,f_1,f_2,\mathcal{W}_{XY},\mathcal{G}_{XY})\}. \tag{7}$$

As a standard HT problem, we are interested in asymptotic behavior of (7). What we get towards this problem is presented in Section 4.

To proceed with definitions, introduce the convex hulls of $\mathcal{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})}$ and $\mathcal{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})}$

$$\widehat{\mathcal{W}}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})} \triangleq \{\sum_{\mathbf{s}\in\mathcal{S}^N}\lambda_{\mathbf{s}}W^N_{\mathbf{s},f_1(\mathbf{X})f_2(\mathbf{Y})},\ 0\leq\lambda_{\mathbf{s}}\leq 1,\ \sum_{\mathbf{s}\in\mathcal{S}^N}\lambda_{\mathbf{s}}=1\}, \tag{8}$$

$$\widehat{\mathcal{G}}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})} \triangleq \{\sum_{\mathbf{s}\in\mathcal{S}^N}\lambda_{\mathbf{s}}G^N_{\mathbf{s},f_1(\mathbf{X})f_2(\mathbf{Y})},\ 0\leq\lambda_{\mathbf{s}}\leq 1,\ \sum_{\mathbf{s}\in\mathcal{S}^N}\lambda_{\mathbf{s}}=1\}. \tag{9}$$

Define for $k=1,2,...$

$$\theta_{\mathrm{AVS}}(R_1,R_2,k,\mathcal{W}_{XY},\mathcal{G}_{XY}) \triangleq \sup_{f_1:\,\log||f_1||\leq kR_1}\ \sup_{f_2:\,\log||f_2||\leq kR_2}\ \min_{\widehat{G}^k_{f(\mathbf{X})\mathbf{Y}}\in\widehat{\mathcal{G}}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}}\ \min_{\widehat{W}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}\in\widehat{\mathcal{W}}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}} \left\{\frac{1}{k}D(\widehat{W}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}||\widehat{G}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})})\right\} \tag{10}$$

where $D(\widehat{W}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}||\widehat{G}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})})$ is the information divergence between two distributions $\widehat{W}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}$ and $\widehat{G}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}$ defined as

$$D(\widehat{W}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}||\widehat{G}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}) = \sum_{\mathbf{x}\in\mathcal{X}^k,\mathbf{y}\in\mathcal{Y}^k}\widehat{W}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}(\mathbf{x},\mathbf{y})\log\frac{\widehat{W}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}(\mathbf{x},\mathbf{y})}{\widehat{G}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}(\mathbf{x},\mathbf{y})}.$$

and

$$\theta_{\mathrm{AVS}}(R_1,R_2,\mathcal{W}_{XY},\mathcal{G}_{XY}) \triangleq \sup_k \theta_{\mathrm{AVS}}(R_1,R_2,k,\mathcal{W}_{XY},\mathcal{G}_{XY}). \tag{11}$$

To deal with special cases of AVCS HT problem with and without data compression we need extra notations.

For the partial data compression case of $R_2>\log|\mathcal{Y}|$, let

$$\beta_{\mathrm{AVS}}(R,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY}),\ \theta_{\mathrm{AVS}}(R,k,\mathcal{W}_{XY}\mathcal{G}_{XY})$$

and $\theta_{\mathrm{AVS}}(R,\mathcal{W}_{XY},\mathcal{G}_{XY})$ be the corresponding version of (7), (10) and (11), respectively, with setting $R_1=R$. Note that

$$\theta_{\mathrm{AVS}}(R,k,\mathcal{W}_{XY},\mathcal{G}_{XY}) = \min_{\widehat{W}^k_{f(\mathbf{X})\mathbf{Y}}\in\widehat{\mathcal{W}}^k_{f(\mathbf{X})\mathbf{Y}}}\ \min_{\widehat{G}^k_{f(\mathbf{X})\mathbf{Y}}\in\widehat{\mathcal{G}}^k_{f(\mathbf{X})\mathbf{Y}}}\theta(R,k,\widehat{W}^k_{f(\mathbf{X})\mathbf{Y}},\widehat{G}^k_{f(\mathbf{X})\mathbf{Y}}) \tag{12}$$

In Fu-Shen [3] AVS HT problem (6) can be rewritten as

$$\beta_{\mathrm{AVS}}(N,\varepsilon,\mathcal{W}_X,\mathcal{G}_X) \triangleq \min_{\mathcal{A}\subset\mathcal{X}^N}\{\overline{\beta}_{\mathrm{AVS}}(\varphi^N,\mathcal{G}_X): \\ \alpha_{\mathrm{AVS}}(\varphi^N,\mathcal{W}_X)\geq 1-\varepsilon,\ \forall\mathbf{s}\in\mathcal{S}^N\}, \tag{13}$$

with $X=Y$ and in Ahlswede-Csiszár [2] as

$$\beta(N,\varepsilon,f,W_{XY},G_{XY}) \triangleq \min_{\mathcal{A}\subset f(\mathcal{X}^N)\times\mathcal{Y}^N}\{G^N_{f(\mathbf{X})\mathbf{Y}}(\mathcal{A}): \\ W^N_{f(\mathbf{X})\mathbf{Y}}(\mathcal{A})\geq 1-\varepsilon,\ \forall\mathbf{s}\in\mathcal{S}^N\}, \tag{14}$$

with consideration

$$\beta(R,N,\varepsilon,W_{XY},G_{XY}) \triangleq \min_{f:\,\log||f||\leq NR}\{\beta(N,\varepsilon,f,W_{XY},G_{XY})\}, \tag{15}$$

where W_{XY} and G_{XY} are unique members of $\mathcal{W}_{XY}$ and $\mathcal{G}_{XY}$, respectively, as $|\mathcal{S}|=1$.

Further, denote for $k=1,2,\ldots$

$$\theta(R,k,W_{XY},G_{XY}) \triangleq \sup_{f:\,\log||f||\leq kR}\{\frac{1}{k}D(W_{f(X)Y}||G_{XY})\}. \tag{16}$$

Section 3 studies this partial compression scenario first.

3. HT WITH COMMUNICATION CONSTRAINTS

The preliminary discussions and definitions above allow to formulate our results. For the AVCS HT problem in Ahlswede-Csiszár scheme we have

Theorem 1. For every $R\geq 0$,

a)

$$\limsup_{N\to\infty}\frac{1}{N}\log\beta_{AVS}(R,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY}) \leq -\theta_{AVS}(R,\mathcal{W}_{\mathbf{XY}},\mathcal{G}_{\mathbf{XY}})\quad\forall\varepsilon\in(0,1),$$

b)

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\beta_{AVS}(R,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY}) \geq -\theta_{AVS}(R,\mathcal{W}_{\mathbf{XY}},\mathcal{G}_{\mathbf{XY}}).$$

Apparently, setting $|\mathcal{S}|=1$ we get the result by Ahlswede and Csiszár [2].

Theorem 2. For every $R\geq 0$ we have

a)

$$\limsup_{N\to\infty}\frac{1}{N}\log\beta(R,N,\varepsilon,W^N_{f(\mathbf{X})\mathbf{Y}},G^N_{f(\mathbf{X})\mathbf{Y}}) \leq -\theta(R,W^N_{f(\mathbf{X})\mathbf{Y}},G^N_{f(\mathbf{X})\mathbf{Y}})\quad\forall\varepsilon\in(0,1),$$

b)

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\beta(R,N,\varepsilon,W^N_{f(\mathbf{X})\mathbf{Y}},G^N_{f(\mathbf{X})\mathbf{Y}})$$
$$\geq -\theta(R,W^N_{f(\mathbf{X})\mathbf{Y}},G^N_{f(\mathbf{X})\mathbf{Y}}).$$

With $R\geq\log|\mathcal{X}|$ we get Stein's lemma for AVS [3].

LEMMA 1 (STEIN'S LEMMA FOR AVS). *For every* $\varepsilon\in(0,1)$ *we have*

$$\limsup_{N\to\infty}-\frac{1}{N}\log\beta_{AVS}(N,\varepsilon,\mathcal{W}_X,\mathcal{G}_X)$$
$$=\min_{\widehat{W}_X\in\widehat{\mathcal{W}}_X}\min_{\widehat{G}_X\in\widehat{\mathcal{G}}_X}D(\widehat{W}_X||\widehat{G}_X),$$

where $\widehat{W}_X$ and $\widehat{G}_X$ are given by (8) and (9) for one source in one dimensional case ($N=1$).

Proof of Theorem 1. A similar proof of Theorem 2 [2] is valid here.

a) Pick a k and consider the k-dimensional problem of HT according to hypotheses

$$H_0:\mathcal{W}^k_{f(\mathbf{X})\mathbf{Y}}\qquad H_1:\mathcal{G}^k_{f(\mathbf{X})\mathbf{Y}}.$$

Applying Stein's lemma for AVS's we get

$$\limsup_{l\to\infty}\left[\frac{1}{lk}\log\beta_{\text{AVS}}(R,lk,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})\right]$$
$$\leq-\theta_{\text{AVS}}(R,k,\mathcal{W}_{XY},\mathcal{G}_{XY})$$

for every $\varepsilon\in(0,1)$. Since $lk\leq N\leq(l+1)k$ we have

$$\beta_{\text{AVS}}(R,(l{+}1)k,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})\leq\beta_{\text{AVS}}(R,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})$$
$$\leq\beta_{\text{AVS}}(R,lk,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY}),$$

it follows that

$$\limsup_{N\to\infty}[\frac{1}{N}\log\beta_{\text{AVS}}(R,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})]$$
$$\leq-\theta_{\text{AVS}}(R,k,\mathcal{W}_{XY},\mathcal{G}_{XY})$$

for every $\varepsilon\in(0,1)$. Since k was arbitrary, this proves assertion a).

To prove the second passage we should perform several estimates.

For $\widehat{G}^N_{f(\mathbf{X})\mathbf{Y}}\in\widehat{\mathcal{G}}^N_{f(\mathbf{X})\mathbf{Y}}$ there exist, $\lambda_{\mathbf{s}}$, $\mathbf{s}\in\mathcal{S}^N$, with properties

$$\sum_{\mathbf{s}\in\mathcal{S}^N}\lambda_{\mathbf{s}}=1,\qquad\text{and}\qquad 0\leq\lambda_{\mathbf{s}}\leq 1,$$

such that

$$\widehat{G}^N_{f(\mathbf{X})\mathbf{Y}}=\sum_{\mathbf{s}\in\mathcal{S}^N}\lambda_{\mathbf{s}}G^N_{\mathbf{s},f(\mathbf{X})\mathbf{Y}}.$$

Then for every $\widehat{G}^N_{f(\mathbf{X})\mathbf{Y}}\in\widehat{\mathcal{G}}^N_{f(\mathbf{X})\mathbf{Y}}$ and $\mathcal{A}\subseteq f(\mathcal{X}^N)\times\mathcal{Y}^N$

$$\widehat{G}^N_{f(\mathbf{X})\mathbf{Y}}(\mathcal{A})\leq\max_{\mathbf{s}\in\mathcal{S}^N}G^N_{\mathbf{s},f(\mathbf{X})\mathbf{Y}}(\mathcal{A}).\tag{17}$$

Similarly, from (17) we have that for every $\mathcal{A}\subseteq f(\mathcal{X}^N)\times\mathcal{Y}^N$ and $\widehat{W}^N_{f(\mathbf{X})\mathbf{Y}}\in\widehat{\mathcal{W}}^N_{f(\mathbf{X})\mathbf{Y}}$

$$\widehat{W}^N_{f(\mathbf{X})\mathbf{Y}}(\mathcal{A}^c)\leq\max_{\mathbf{s}\in\mathcal{S}^N}W^N_{\mathbf{s},f(\mathbf{X})\mathbf{Y}}(\mathcal{A}^c).$$

For a fixed $\widehat{W}^N_{f(\mathbf{X})\mathbf{Y}}\in\widehat{\mathcal{W}}^N_{f(\mathbf{X})\mathbf{Y}}$

$$\{\mathcal{A}:\ \mathcal{A}\subseteq f(\mathcal{X}^N)\times\mathcal{Y}^N\,\&\,W^N_{\mathbf{s},f(\mathbf{X})\mathbf{Y}}(\mathcal{A})\geq 1-\varepsilon,\ \forall\mathbf{s}\in\mathcal{S}^N\}$$
$$=\{\mathcal{A}:\ \mathcal{A}\subseteq f(\mathcal{X}^N)\times\mathcal{Y}^N\ \&\ W^N_{\mathbf{s},f(\mathbf{X})\mathbf{Y}}(\mathcal{A}^c)\leq\varepsilon,\ \forall\mathbf{s}\in\mathcal{S}^N\}$$
$$\subseteq\{\mathcal{A}:\ \mathcal{A}\subseteq f(\mathcal{X}^N)\times\mathcal{Y}^N\ \&\ \widehat{W}^N_{f(\mathbf{X})\mathbf{Y}}(\mathcal{A}^c)\leq\varepsilon\}$$
$$=\{\mathcal{A}:\ \mathcal{A}\subseteq f(\mathcal{X}^N)\times\mathcal{Y}^N\ \&\ \widehat{W}^N_{f(\mathbf{X})\mathbf{Y}}(\mathcal{A})\geq 1-\varepsilon\}.\tag{18}$$

From (17) and (18) we have for every $\widehat{W}^N_{f(\mathbf{X})\mathbf{Y}}\in\widehat{\mathcal{W}}^N_{f(\mathbf{X})\mathbf{Y}}$ and $\widehat{G}^N_{f(\mathbf{X})\mathbf{Y}}\in\widehat{\mathcal{G}}^N_{f(\mathbf{X})\mathbf{Y}}$

$$\beta_{\text{AVS}}(R,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})$$
$$\geq\min_{\mathcal{A}\subseteq f(\mathcal{X}^N)\times\mathcal{Y}^N,\,\widehat{W}^N_{f(\mathbf{X})\mathbf{Y}}(\mathcal{A})\geq 1-\varepsilon}\widehat{G}^N_{f(\mathbf{X})\mathbf{Y}}(\mathcal{A}),$$
$$\geq\min_{f:\,\log||f||\leq NR}\ \min_{\mathcal{A}\subseteq f(\mathcal{X}^N)\times\mathcal{Y}^N,\,\widehat{W}^N_{f(\mathbf{X})\mathbf{Y}}(\mathcal{A})\geq 1-\varepsilon}\widehat{G}^N_{f(\mathbf{X})\mathbf{Y}}(\mathcal{A})$$
$$=\beta(R,N,\varepsilon,\widehat{W}^N_{f(\mathbf{X})\mathbf{Y}},\widehat{G}^N_{f(\mathbf{X})\mathbf{Y}}).\tag{19}$$

b) In view of (12), (19) and the **b**) point of Theorem 2 we get the second part of Theorem 1.

We would like to prove a stronger result (as it is done in [2]) conjectured here as

Theorem 3.

$$\lim_{N\to\infty}\frac{1}{N}\log\beta_{AVS}(R,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})$$
$$=-\theta_{AVS}(R,\mathcal{W}_{\mathbf{XY}},\mathcal{G}_{\mathbf{XY}})$$

for all $R\geq 0$ *and* $\varepsilon\in(0,1)$.

Remark 1. *The single-letter characterization problem of (6) remains open.*

4. HT WITH MULTITERMINAL DATA COMPRESSION

For our extension of Han's [1] HT problem with multiterminal data compression was possible to get the following results.

Theorem 4. *For every* $R_1\geq 0$*,* $R_2\geq 0$ *and* $\forall\varepsilon\in(0,1)$ *we have*

a)

$$\limsup_{N\to\infty}\left[\frac{1}{N}\log\beta_{AVS}(R_1,R_2,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})\right]$$
$$\leq-\theta_{AVS}(R_1,R_2,\mathcal{W}_{XY},\mathcal{G}_{XY}),$$

b)

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\left[\frac{1}{N}\log\beta_{AVS}(R_1,R_2,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})\right]$$
$$\geq-\theta_{AVS}(R_1,R_2,\mathcal{W}_{XY},\mathcal{G}_{XY}),$$

c)

$$\limsup_{N\to\infty}\left[\frac{1}{N}\log\beta_{AVS}(R_1,R_2,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})\right]$$
$$\geq-\min_{\widehat{W}^N_{XY}\in\widehat{\mathcal{W}}^N_{XY}}\min_{\widehat{G}^N_{XY}\in\widehat{\mathcal{G}}^N_{XY}}D(\widehat{W}^N_{XY}||\widehat{G}^N_{XY}).$$

Proof of Theorem 4. a) Pick a k and consider the k-dimensional problem of HT according to hypotheses

$$H_0 : \mathcal{W}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})} \qquad H_1 : \mathcal{G}^k_{f_1(\mathbf{X})f_2(\mathbf{Y})}.$$

Applying Stein's lemma for AVS's we get

$$\limsup_{l\to\infty}\left[\frac{1}{lk}\log\beta_{\text{AVS}}(R_1,R_2,lk,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})\right]$$

$$\leq -\theta_{\text{AVS}}(R_1,R_2,k,\mathcal{W}_{XY},\mathcal{G}_{XY})$$

for every $\varepsilon\in(0,1)$. Since $lk \leq N \leq (l+1)k$ we have

$$\beta_{\text{AVS}}(R_1,R_2,(l+1)k,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})$$

$$\leq \beta_{\text{AVS}}(R_1,R_2,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})$$

$$\leq \beta_{\text{AVS}}(R_1,R_2,lk,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY}),$$

then it follows that

$$\lim\sup_{N\to\infty}[\frac{1}{N}\log\beta_{\text{AVS}}(R_1,R_2,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})]$$

$$\leq -\theta_{\text{AVS}}(R_1,R_2,k,\mathcal{W}_{XY},\mathcal{G}_{XY})$$

for every $\varepsilon\in(0,1)$. Since k was arbitrary, this proves the a) point of the theorem.

b) By analogy of (19)

$$\beta_{\text{AVS}}(R_1,R_2,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})$$

$$\geq \beta(R_1,R_2,N,\varepsilon,\widehat{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})},\widehat{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})}). \tag{20}$$

For every function f_1, f_2 defined on $\mathcal{X}^N$ and every $\mathcal{A}\subset\mathcal{L}^1_N\times\mathcal{L}^2_N$, we have

$$D(\widehat{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})}||\widehat{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})}) \geq \alpha\log\frac{\alpha}{\beta}+(1-\alpha)\log\frac{1-\alpha}{1-\beta} \tag{21}$$

where

$$\alpha=\widehat{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})}(\mathcal{A}), \qquad \beta=\widehat{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})}.$$

By (6) and (7) we can choose f_1, f_2 and $\mathcal{A}$ such that

$$\log||f_l||\leq NR_l, \qquad \alpha\geq 1-\varepsilon$$

and

$$\beta=\beta(R_1,R_2,N,\varepsilon,\widehat{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})},\widehat{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})}).$$

Then (10), (11) and (21) give

$$\theta(R_1,R_2,\widehat{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})},\widehat{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})})$$

$$\geq \theta(R_1,R_2,N,\widehat{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})},\widehat{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})})$$

$$\geq \frac{1}{N}D(\widehat{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})}||\widehat{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})})$$

$$\geq -\frac{1-\varepsilon}{N}\log\beta(R_1,R_2,N,\varepsilon,\widehat{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})},\widehat{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})})-\frac{h(\alpha)}{N}$$

where

$$h(\alpha)=-\alpha\log\alpha-(1-\alpha)\log(1-\alpha).$$

Then for every $\widehat{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})}$, $\widehat{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})}$

$$\frac{1-\varepsilon}{N}\log\beta(R_1,R_2,N,\varepsilon,\widehat{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})},\widehat{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})})$$

$$\geq -\theta(R_1,R_2,\widehat{W}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})},\widehat{G}^N_{f_1(\mathbf{X})f_2(\mathbf{Y})})$$

$$\geq -\theta_{\text{AVS}}(R_1,R_2,k,\mathcal{W}_{XY},\mathcal{G}_{XY}).$$

(20) and the last inequality complete the proof.

c) Using Stein's lemma for AVS's (Lemma 1) and the monotonicity of function $\beta_{\text{AVS}}(R,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})$ in R, as well as the same property for $\beta_{\text{AVS}}(R_1,R_2,N,\varepsilon,\mathcal{W}_{XY},\mathcal{G}_{XY})$ in (R_1,R_2), we get the point c) of the theorem.

Remark 2. $R_2\to\infty$ we get Theorem 1 from the points a) and b) of Theorem 4.

In case of $|\mathcal{S}|=1$ Theorem 4 results in an answer to Han's HT problem as

Theorem 5. For every $R_1\geq 0$, $R_2\geq 0$ and $\forall\varepsilon\in(0,1)$ we have

a)

$$\limsup_{N\to\infty}\left[\frac{1}{N}\log\beta(R_1,R_2,N,\varepsilon,W_{XY},G_{XY})\right]$$

$$\leq -\theta(R_1,R_2,W_{XY},G_{XY}),$$

b)

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\left[\frac{1}{N}\log\beta(R_1,R_2,N,\varepsilon,W_{XY},G_{XY})\right]$$

$$\geq -\theta(R_1,R_2,W_{XY},G_{XY}),$$

c)

$$\limsup_{N\to\infty}\left[\frac{1}{N}\log\beta(R_1,R_2,N,\varepsilon,W_{XY},G_{XY})\right]$$

$$\geq -D(\widehat{W}_{XY}||\widehat{G}_{XY}),$$

$\beta(R_1,R_2,N,\varepsilon,W_{XY},G_{XY})$ and $\theta(R_1,R_2,W_{XY},G_{XY})$ being respective notations specialized from (7) and (11) for the model [1].

Remark 3. Comparison of Theorem 5 with the single- letter estimate given in [1] for the HT with multiterminal data compression is another open problem.

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