On Layout Problems of Kneser Graphs

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ABSTRACT

The minimal linear arrangement problem (MinLA) is defined as follows: given a graph G, find a linear ordering for the vertices of G on a line such that the sum of the edge lengths is minimized over all orderings. The another graph layout problem, CUTWIDTH, asks, given a graph G, and a positive integer k, whether there exists a linear ordering of the vertices of G so that any line inserted between two consecutive vertices of the layout cuts (intersects with) at most k edges. The CUTWIDTH of the input graph is the smallest integer for which the question can be answered positively. In this paper a numbering is introduced for the Kneser graph K(n,r) when r=2 and proved that it is optimal both for MinLA and CUTWIDTH.

Keywords

MinLA, CUTWIDTH, Kneser graphs.

1. INTRODUCTION

We consider undirected finite graphs with no loops or multiple edges. For a graph G=(V,E), we denote its vertex and edge set by V and E, respectively, with p=|V| and q=|E|. In the following all undefined graph-theoretical terms can be found in [1].

Consider two graph layout problems. Given a graph G=(V,E), a layout L is a one-to-one mapping L : V → {1,…,p}. For a given G=(V,E) and a layout L, set

$$LA(G,L)=\sum_{(u,v)\in E}|L(u)-L(v)|$$

An optimal linear arrangement of G is a layout which provides the minimum for the LA(G,L). We denote $\mathrm{LA(G)}=\min_L LA(G,L)$. LA(G) is also known as a wirelength of a graph G.

For a given G=(V,E) and a subset A ⊆V denote *I(A)*={(u,v) ∈E|u,v∈A} and *θ(A)*={(u,v) ∈E|u∈A, v∉A}. The first set includes all edges of the graph which both sides are in A, while the second set includes those edges which exactly one side is in A. We say that a subset A is optimal with respect to the function *I* (*θ*) if it provides the maximum (minimum) for |*I*(A)| (respectively for |*θ*(A)|), taken over all subsets of V of the cardinality |A|.

For a given G=(V,E) and a layout L, denote V_t^L= {v∈ V|L(v) ≤ t}. We call V_t^L an initial segment with respect to layout L. The CUTWIDTH of a layout L is defined as CW(G,L) = $\max_{1\le t\le p-1} |\theta(V_t^L)|$ and its minimum over all layouts – as the cutwidth of G: CW(G) = $\min_L CW(G,L)$.

These two important graph layout problems were first proposed as models in circuit design ([2]), and more recently they have found applications in areas like protein engineering ([3],[4]). Unfortunately both problems are NP-complete ([5]) and they remain NP-complete even for certain classes of graphs. The MinLA is NP-complete for bipartite graphs and even for interval graphs. The CUTWIDTH is NP-complete for planar graphs with maximum degree 3, unit disk graphs, split graphs. On the other hand polynomial-time algorithms for the exact computation of these problems are known only for very few graph classes. We refer the reader to [6] for a survey of known results on the MinLA, CUTWIDTH and other graph layout problems.

Harper in [7] introduced another expression for the LA(G,L):

$$LA(G,L)=\sum_{(u,v)\in E}|L(u)-L(v)|=\sum_{t=1}^{p-1}\theta(V_t^L) \qquad (1)$$

From this equation it is clear to see that if there is a layout for which all initial segments V_t^L are optimal with respect to *θ,* then such a layout will provide a minimum both for MinLA and CUTWIDTH. Not all graphs have such nice property (known as a nested solution [7]). For example, the binary n-cube ([7]), complete n-partite graphs ([8]) permit such ordering, but rectangular grids, torus - not ([9]). In this paper we show that the Kneser graph K(n,k) has a nested solution for k=2, and does not permit it for k>2 in general.

2. Layout Problems for Kneser Graphs

The Kneser graph K(n,r) (r < n/2) is the graph whose vertices are all subsets of the set {1,2,…, n} with the cardinality r, and two vertices are connected by an edge if and only if the corresponding subsets do not intersect. So K(n,r) has C_n^r vertices and is a regular graph with the vertex degree C_{n-r}^r. For example K(5,2) is the Petersen graph. It is easy to see that K(n,2) is the complement of the line graph of K_n. We will use this consideration in the proofs. Notice that the line graph of a graph G (denoted by $\mathcal{L}$(G)) is a graph which vertices correspond to the edges of G and two vertices of $\mathcal{L}$(G) are connected by an edge if and only if the corresponding them edges in G are adjacent.

For the simplicity let's label vertices of K_n by numbers 1,2,…,n. Then the vertices of K(n,2) can be represented as pairs (i,j), where i < j, and i,j∈{1,2,…,n}. The vertices (i,j) and (s,t) are incident if and only if all i,j,s,t are different.

For a subset of vertices A of the K(n,2), denote

$\beta_r(A)$=|{(i,j)/ (i,j)∈A; i=r or j=r, i,j,r=1,2,…,n}|.

Actually $\beta_r(A)$ is the number of vertices (i,j) ∈ A which corresponding edges in K_n are adjacent to the vertex r.

Let K(n,2)=(V,E) and |V|=p,|E|=q. The following lemma gives the complete description of optimal subsets of the K(n,2).

Lemma. A subset A∈V is optimal with respect to function *I* if and only if |β_i(A) - β_j(A)|≤ 1 for all i,j =1,2,…,n.

Proof. Consider a subset of vertices A∈V and let |A|=m. We are going to represent *I*(A) via m and β_i(A)-s. Denote by Ω the spanning graph of K_n which includes only the edges corresponding to the vertices of A. It is easy to see that the vertices of Ω have the degrees β_1(A), β_2(A),…,β_n(A). Note that in general some β_i(A) can be 0.

We will take an advantage of the Theorem 8.1 from the book of Harary ([1]), which stated that if G is a graph with p vertices and q edges and with vertex degrees $d_1,d_2,\ldots d_p$, then its line graph $\mathcal{L}$(G) has q vertices and $\frac{1}{2}\cdot\sum_{i=1}^{p} d_i^2 - q$ edges.

Following to this theorem the edge number of $\mathcal{L}(\Omega)$ equals $\frac{1}{2}\cdot\sum_{i=1}^{m}\beta_i^2(A) - m$. Consequently the edge number of its complement, i.e. of the subgraph induced by the vertices of A, equals:

$$\frac{m\cdot(m-1)}{2} - (\frac{1}{2}\cdot\sum_{i=1}^{m}\beta_i^2(A) - m) = \frac{m\cdot(m+1)}{2} - \frac{1}{2}\cdot\sum_{i=1}^{m}\beta_i^2(A),$$

which is just I(A), and to maximize it, we have to solve a simple optimization problem: minimize $\sum_{i=1}^{m}\beta_i^2(A)$ with constraints $\beta_i \geq 0$, $\sum_{i=1}^{m}\beta_i(A) = 2\cdot m$. Its integer-valued solution obviously is $|\beta_i(A) - \beta_j(A)| \leq 1$ for all i,j =1,2,…,n. ■

It remains to construct a layout for the K(n,2), where each initial segment possesses the property described in Lemma. Here we will use factorizations of K_n.

R-factorization of a graph is its decomposition into edge disjoint subgraphs (R-factors), in which each vertex is the endpoint of R edges. A graph is said to be R-factorable if it admits an R-factorization. In particular, 1-factor is a collection of disjoint edges, which together are incident on all vertices of the graph (also called – perfect matching). 2-factor is a collection of edges forming one or more disjoint circles which cover all vertices of the graph. We are interested in 1-factorization of K_n when n is even and in its 2-factorization for the odd n.

We will use two results from the book of Harary ([1]).

Theorem 9.1. For an even n, K_n admits a 1-factorization.

*Theorem 9.6. For an odd n, K_n admits a 2-factorization, where each 2-factor is a spanning (**Hamiltonian)** circle.*

Consider a following *f*-ordering of K(n,2). First note that the proofs of above theorems are constructive and one can assume that factors are available.

n is even. Let F1,F2,…,Fn-1 are 1-factors of K_n and L(F1), L(F2)…, L(Fn-1) are corresponding to them vertex subsets of K(n,2). Obviously subgraphs induced by L(Fi)-s are cliques on $\frac{n}{2}$ vertices. Then in f-ordering L(Fi)-s are ordered one by one and occupy continuous segments of numbers while in each segment vertices of L(Fi)-s are ordered arbitrarily.

n is odd. Let $C_1,C_2,\ldots,C_{(n-1)/2}$ are 2-factors (Hamiltonian circles of length n) of K_n and $\mathcal{L}(C_1),\mathcal{L}(C_2),\ldots,\mathcal{L}(C_{n-1})$ are corresponding to them vertex subsets of K(n,2). In *f*-ordering again $\mathcal{L}(C_i)$-s are ordered one by one and occupy continuous segments of numbers. However $\mathcal{L}(C_i)$-s require a special ordering. A circle consisting of vertices $v_1,v_2,\ldots v_n$ and edges (v_1,v_n) and (v_i,v_{i+1}) for i=1,2,…,n-1, is ordered as the following: $v_1,v_3,\ldots v_{n-2}, v_2,v_4,\ldots v_{n-1},v_n$.

It is easy to verify that for the *f*-ordering of K(n,2), each initial segment satisfies conditions of Lemma, i.e. is optimal with respect to the function *I*. It is easy to show that for regular graphs optimal subsets for the functions *I* and θ are the same: a subset is optimal with respect to the function *I* if and only if it is optimal with respect to θ. So each initial segment is optimal with respect to the function θ too. Hence with the equation (1) we can prove the following:

Theorem-1. *f*-layout is optimal both for MinLA and CUTWIDTH problems for K(n,2).

Unfortunately for r>2 the Kneser graphs K(n,r) in general have not nested solutions. For the sake of simplicity we will show this for n=r(r+1).

Theorem-2. K(r(r+1),r) has not a nested solution.

Proof. Maximal cliques of K(r(r+1),r) have the cardinality r+1 and if the graph has a nested solution L, then obviously its each initial segment V_t^L (t ≤ r+1) should be a clique (to maximize $I(V_t^L)$. Moreover, we will show that in this case the initial segment $V_{r(r+1)}^L$ should consist of r maximal cliques occupying continuous segments of numbers.

Proposition. If L is a nested solution for K(r(r+1),r) then $V_{r(r+1)}^L$ consists of r maximal cliques occupying continuous segments of numbers.

Proof. It is easy to see that any vertex which does not belong to some maximal clique K_{r+1}, can have at most r-1 neighbors there. Next it is possible to construct r maximal cliques where any vertex from a clique has exactly r-1 neighbors from each other cliques. Let the first clique is

[1,2,…,r+1], [r+2,…,2r+2],…,[(r-1)(r+1)+1,…,r(r+1)],

the second is shifted cyclically on one position:

[2,3,…,r+2], [r+3,…,2r+3],…,[(r-1)(r+1)+2,…,r(r+1),1],

etc. The last, r-th clique is shifted cyclically from the previous (r-1)-th clique on one position:

[r+1,r+2,…,2r+1], [2r+2,…,3r+2],…,[r(r+1),1,2,…,r].

It is easy to see that in this construction any vertex from a clique has exactly r-1 neighbors from each other clique.

The proposition can be proved using the usual mathematical induction method. Let t is an integer from the interval [1; r(r+1)-1]. The proposition obviously is true for t ≤ r+1. Let it is true for some t = s(r+1)+z, where 1 ≤s ≤r-1 and 0≤z ≤r. By the inductive assumption, V_t^L consists of s maximal cliques and a z-clique, all occupying continuous segments of numbers. Any vertex which does not belong to V_t^L can have at most r-1 neighbors from maximal cliques of V_t^L. So a vertex x, which has exactly r-1 neighbors and besides it is incident to all vertices of the z-clique of V_t^L will provide the maximal number of inner edges for the subgraph induced by the vertex set $V_t^L \cup \{x\}$. By the above construction there exists a vertex with such properties. ■

Note that $|I(V_{r(r+1)}^L)|$ =(r^2-r+1)r(r+1)/2. Let's show that $V_{r(r+1)}^L$ is not optimal. The contradictory sample A containing r(r+1) vertices can be constructed as the following. Let's partition the set {1,2,…,r(r+1)} into disjoint subsets: {1,2,…,r+1}, {r+2,…,2r+2},…,{(r-1)(r+1)+1,…,r(r+1)}, and take from each of them all its r-subsets. The subgraph induced by these vertices form a complete r-partite graph on r(r+1) vertices which has (r^2-1)r(r+1)/2= $I(A)$ edges, which is greater than $|I(V_{r(r+1)}^L)|$ =(r^2-r+1)r(r+1)/2 for r>2. ■

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