On the existence of the *tt*-mitotic low set which is not *btt*-mitotic

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ABSTRACT

Let us adduce some definitions:

A *splitting* of noncomputable computably enumerable (c.e.) set A is a pair A_0, A_1 of disjoint c.e. sets such that $A_0 \cup A_1 = A$. An c.e. set A is *tt-mitotic* (*btt-mitotic*) if there is a splitting A_0, A_1 of A such that $A_0 \equiv_{tt} A_1 \equiv_{tt} A$ ($A_0 \equiv_{btt} A_1 \equiv_{btt} A$).

In this paper it is proved, that there exists a *tt*-mitotic low set, which is not *btt*-mitotic.

Keywords

Computably enumerable (c.e.) set, mitotic set, low degree, *btt*-reducibility, *tt*-reducibility.

Notation.

We shall use notions and terminology introduced in [1], [4], [5].

Let $<x, y>$ denote the image of $<x, y>$ under the standard pairing function $\frac{1}{2}\left\{x^2 + 2xy + y^2 + 3x + y\right\}$, which is a 1:1 computable function from $\omega \times \omega$ onto ω.

Let π_1 and π_2 denote the inverse functions $\pi_1(<x, y>) = x$ and $\pi_2(<x, y>) = y$. Let $<x_1, x_2, x_3>$ denote $<<x_1, x_2>, x_3>$ and $<x_1, x_2, \ldots, x_n>$ denote $<\ldots<<x_1, x_2>, x_3>, \ldots, x_n>$.

For any set $A \subseteq \omega$ define the y-*column* of A.

$A^{(y)} = \left\{<x, z>: <x, z> \in A \;\&\; z = y\right\}$.

$M_{y,x} = \omega^{(<y,x>)}$.

$M_e^0 = \bigcup_{i=0}^{\infty} M_{e,2i}$; $M_e^1 = \bigcup_{i=0}^{\infty} M_{e,2i+1}$.

$M^0 = \bigcup_{e=0}^{\infty} M_e^0$; $M^1 = \bigcup_{e=0}^{\infty} M_e^1$.

$\widetilde{M}_{e,i} = M_{e,2i} \cup M_{e,2i+1}$; $M_e = \bigcup_{i=0}^{\infty} M_{e,i} = \bigcup_{i=0}^{\infty} \widetilde{M}_{e,i}$.

Thus, $M^0 \cup M^1$ ω
=.

The definitions of tt- and btt- reducibilities are from [4].

$\varphi(x)\downarrow$ denotes, that $\varphi(x)$ is defined, and $\varphi(x)\uparrow$ denotes, that $\varphi(x)$ is undefined.

Definitions. A T-degree $\boldsymbol{a} \le \boldsymbol{0}'$ is *low* if $\boldsymbol{a}' = \boldsymbol{0}'$ (i.e. if the jump $\boldsymbol{a}'$ has the lowest degree possible).

A set A is *low*, if $\deg_T(A)$ is low.

The order pair **<<** $x_1, \cdots, x_k$ **>,** α **>**, where **<** $x_1, \cdots, x_k$ **>** is a k-tuple of integers and α is a k-ary Boolean function ($k > 0$) is called a *truth-table condition* (or *tt-condition*) *of norm* k. The set $\{x_1, \cdots, x_k\}$ is called the *associated set of the tt-condition*.

The tt-condition **<<** $x_1, \cdots, x_k$ **>,**α **>**, is *satisfied* by A if $(\alpha c_A(x_1), \cdots, c_A(x_k)) = 1$, where c_A is a characteristic function for A.

Each tt-condition is a finite object; clearly an effective coding can be chosen which maps all tt-conditions (of varying norm) onto ω.

Assume henceforth that a particular such coding has been chosen. Where we speak of "tt-condition x", we shall mean the tt-condition with the code number x.

Notation. *Code* **<<**$x_1, \cdots, x_k$ **>,**α **>** denotes the code number of tt-condition **<<**$x_1, \cdots, x_k$ **>,**α **>** in this coding**.**

Definitions. A is *truth-table reducible to* B (notation: $A \le_{tt} B$) if there is a computable function f such that for all x, [$x \in A \Leftrightarrow tt$-condition $f(x)$ is satisfied by B]. We also abbreviate "truth-table reducibility" as "tt-reducibility".

A is *bounded truth-table reducible to* B (notation: $A \le_{btt} B$) if ($\exists$ computable f) ($\exists m$)($\forall x$) [tt-condition $f(x)$ has norm $\le m$, and $[x \in A \Leftrightarrow f(x)$ is satisfied by B]].

We abbreviate "bounded truth-table reducibility" as "btt-reducibility".

Let $A \leq_{tt} B$ and $(\forall x)$ [$x \in A \Leftrightarrow tt$ -condition $f(x)$ be satisfied by B] and $\varphi_n = f$. Then we say that $A \leq_{tt} B$ *by* φ_n.

Let us modify notations defined in [3] with the purpose to adapt them to our theorem.

Let h be a computable function from ω onto ω^5. Define $(Y_i, Z_i, \vartheta_i, \psi_i, j_i)$ to be a quintuple $(W_{i_0}, W_{i_1}, \varphi_{i_2}, \varphi_{i_3}, i_4)$, where $h(i) = (i_0, i_1, i_2, i_3, i_4)$.

If A is c.e. then we say that the *non- btt -mitotic condition of i order is satisfied for A*, if it is not the case that

i) Y_i, Z_i is a splitting of A &

ii) $A \leq_{btt} Y_i$ with norm p (where $p = \pi_1(j_i)$) &

iii) $A \leq_{btt} Z_i$ with norm q (where $q = \pi_2(j_i)$).

Notation. $u^{tt}(i,n,s) = \begin{cases} x^i_{k_n}, & \text{if } \varphi_{i,s}(n)\downarrow, \\ 0, & \text{otherwise} \end{cases}$,

where *tt*-condition $\varphi_i(n) = \ll x^i_1, \cdots, x^i_{k_n} \gg, \alpha^i_n >$.

We define two computable functions that will be of use later.

1. $k(i,n,s) = \max\{ \{u^{tt}(i_2,m,s) : m \leq n\} \cup \{u^{tt}(i_3,m,s) : m \leq n\} \}$.

2. $L(A,i,s) = \mu n[\neg(c_A(n) = 1 \Leftrightarrow tt$ - condition $\vartheta_i(n)$ with norm p satisfied by $Y_i) \vee \neg(c_A(n) = 1 \Leftrightarrow tt$ - condition $\psi_i(n)$ with norm q satisfied by $Z_i)]$, where $h(i) = (i_0, i_1, i_2, i_3, i_4),\ \pi_1(i_4) = p,\ \pi_2(i_4) = q$.

$(Y_i, Z_i, \vartheta_i, \psi_i, j_i)$ is *btt -threatening A through x at stage s* if the following hold:

i) $i \leq s$,

ii) $x < L(A,i,s)$,

iii) $Y_i^s \cap Z_i^s = \varnothing$,

iv) $c_A^s(m) = (Y_i^s \cup Z_i^s)(m)$ for all $m \leq k(i,x,s)$,

v) $(\forall y \leq x)(\vartheta_i^s(y)\downarrow \ \&\ \psi_i^s(y)\downarrow)\ \&\ (\forall y \leq x)$ [the norm of $\vartheta_i^s(y)$ is less or equal than p and the norm of $\psi_i(y)$ is less or equal than q], where $h(i) = (i_0, i_1, i_2, i_3, i_4),\ \pi_1(i_4) = p,\ \pi_2(i_4) = q$.

Let us prove the following theorem.

Theorem. There exists a tt -mitotic low set, which is not btt -mitotic.

Proof (sketch).
This is proved using a finite injury priority argument. We construct a set A in stages s, $A = \bigcup_{s\in\omega} A_s$ such, that $\deg_T(A) = \mathbf{0}'$ and A is not btt -mitotic.

Construct A, to satisfy, for all $e \in \omega$, the requirements:

N_e : $\Phi_e(A)(e)\downarrow$ has limit in s, the stage.

R_e : The non-*btt*-mitotic condition of order e is satisfied for A.

We also ensure that A is tt -mitotic.

Notation. R_i *requires attention at stage s* if there exists such x that $(Y_i, Z_i, \vartheta_i, \psi_i, j_i)$ is *btt -threatening A through x at stage s* and if it is not *satisfied.*

N_e *requires attention at stage s* if $\Phi_{e,s}(A_s)(e)\downarrow$ and if it is not *satisfied.*

Let $a_0, a_1, \ldots, a_n, \ldots$ be the members of set A in increasing order. Denote $id(A)(x) = a_x$.

Let $h(e) = (e_0, e_1, e_2, e_3, e_4)$, $\pi_1(e_4) = p,\ \pi_2(e_4) = q$.

For any e, k define

$M^*_{e,2k} = \{ id(M_{e,2k})(1), id(M_{e,2k})(2), \ldots, id(M_{e,2k})(p+q+1) \}$,

$M^*_{e,2k+1} = \{ id(M_{e,2k+1})(0), id(M_{e,2k+1})(1), \ldots, id(M_{e,2k+1})(p+q) \}$.

Let $x(e,s)$ be the *follower* associated with requirement R_e (at stage s).

Let R_e requires attention at stage s (through $x(e,s)$).

In this case $\vartheta_e^s(x(e,s))\downarrow$ & $\psi_e^s(x(e,s))\downarrow$.

Let $as^2(e,s)$ denote the associated set of tt -condition $\vartheta_e(x(e,s))$; $as^3(e,s)$ denote the associated set of tt -condition $\psi_e(x(e,s))$; $as^*(e,s)$ denote the set $as^2(e,s) \cup as^3(e,s)$.

Order the requirenments in the following priority ranking: $N_0, R_0, N_1, R_1, N_2, R_2, \ldots$.

Construction.

Stage $s = 0$: Let $A_0 = \varnothing$,

$x(e,0) = id(M_{e,o})(0)$ for all e.

Stage $s+1$: Act on the highest priority requirement which requires attention, if such a requirement exists:

If N_e requires attention, then set $x(\hat{e}, s+1) = id(M_{\hat{e},2s})(0)$ for all $\hat{e} \geq e$. This prevents injury to N_e by lower priority unsatisfied.requirements as we assume that s bounds the use of halting computation. Declare N_e satisfied; declare all lower *R,N* unsatisfied.

Let R_e require attention at stage s (through $x(e,s)$).

Let $x(e,s) \in M_{e,2k}$ (that is $x(e,s) \in id(M_{e,2k})(0)$). Find z such, that

$[\, id(M^*_{e,2k+1})(z) \notin as^*(e,s)$ & $id(M^*_{e,2k})(z) \notin as^*(e,s)\,]$.

There exists such an integer z.

Set $A_{s+1} = A_s \cup \{x(e,s)\} \cup \{id(M^*_{e,2k})(z)\} \cup \{id(M^*_{e,2k+1})(z)\}$.

Set $x(\hat{e}, s+1) = id(M_{\hat{e},2s})(0)$ for all $\hat{e} \geq e$.

Declare R_e satisfied, declare all lower *R,N* unsatisfied.

Let us prove that $\widetilde{A} \equiv_{tt} \widetilde{\widetilde{A}}$ (where $\widetilde{A} = A \cap M^0$, $\widetilde{\widetilde{A}} = A \cap M^1$).

We must construct the function g_0 which tt -reduces $\widetilde{A}$ to $\widetilde{\widetilde{A}}$ and the function g_1 which tt -reduces $\widetilde{\widetilde{A}}$ to $\widetilde{A}$.

In this case there would exist computable functions $\widetilde{g}_0, \widetilde{g}_1$ such that $A \leq_{tt} \widetilde{A}$ by function $\widetilde{g}_0$ and $A \leq_{tt} \widetilde{\widetilde{A}}$ by function $\widetilde{g}_1$, because M^0, M^1 are computable sets.
We shall construct the functions g_0, g_1 proceeding from the following considerations.

Construction of g_0:

We shall show how to compute $g_0(x)$ for any x.
If $\neg \exists e,k\ (x \in M^*_{e,2k})$, then define
$g_0(x) = code$ **<<** x **>,** α_0 **>,** where $\alpha_0(x) = 0$ for all $x \in \{0,1\}$ **.**
If exist e,k such, that $x \in M^*_{e,2k}$ (where $e = < i_0, i_1, i_2, i_3, i_4 >$, $\pi_1(i_4) = p,\ \pi_2(i_4) = q$ for some $i_0, i_1, i_2, i_3, i_4, p, q$), then there are two possible cases:

i) If $x = id(M^*_{e,2k})(0)$, then define $g_0(x) = code$ **<<** $id(M_{e,2k+1})(0), id(M_{e,2k+1})(1), \ldots, id(M_{e,2k+1})(p+q)$ **>,** α_1 **>,**
where $\alpha_1(x_0, x_1, \ldots, x_{p+q}) = \begin{cases} 0, & \text{if } x_0 = x_1 = \ldots = x_{p+q} = 0 \\ 1, & \text{otherwise.} \end{cases}$.

ii) If $x \neq id(M^*_{e,2k})(0)$ and $x \in M^*_{e,2k}$, then find z such that $x = id(M^*_{e,2k})(z)$.

Now define $g_0(x) = code$ **<<** $id(M^*_{e,2k+1})(z),$ **>,** α_2 **>,** where $\alpha_2(x) = x$ for all $x \in \{0,1\}$ **.**

Construction of g_1:

We shall show how to compute $g_1(x)$ for any x.
If $\neg \exists e,k\ (x \in M^*_{e,2k+1})$, then define
$g_0(x) = code$ **<<** x **>,** α_0 **>,** where $\alpha_0(x) = 0$ for all $x \in \{0,1\}$.
If $\exists e,k\ (x \in M^*_{e,2k+1})$, then find z such, that .
Now define
$g_1(x) = code$ **<<** $id(M^*_{e,2k})(z)$ **>,** α_2 **>,** where $\alpha_2(x) = x$ for all $x \in \{0,1\}$ **.**

The functions g_0, g_1 satisfy the abovementioned requirements. □

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