**Computing De Morgan and Quasi-De Morgan Functions**

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**ABSTRACT**  
In this paper we give various algorithms for computation of De Morgan and quasi-De Morgan functions. We define the disjunctive and conjunctive normal forms for De Morgan and quasi-De Morgan functions and introduce the concept of Zhegalkin polynomial for the case of quasi-De Morgan functions as well as prove some results analogous to the Zhegalkin representation theorem on Boolean functions.

**Keywords**  
Antichain, Boolean function, monotone Boolean function, De Morgan and quasi-De Morgan function, disjunctive (conjunctive) normal form, Zhegalkin-type representation.

1. INTRODUCTION AND PRELIMINARIES

An algebra \((Q; \{+,-,\cdot,0,1\})\) with two binary, one unary and two nullary operations is called a **Boolean algebra** if \((Q; \{+,-,0,1\})\) is a bounded distributive lattice with least element 0 and greatest element 1 and the algebra \((Q; \{+,-,\cdot,0,1\})\) satisfies the following identities:  
\[ x + x' = 1, \quad x \cdot x' = 0. \]

An algebra \((Q; \{+,-,\cdot,0,1\})\) with two binary, one unary and two nullary operations is called a **De Morgan algebra** if \((Q; \{+,-,0,1\})\) is a bounded distributive lattice with least element 0 and greatest element 1 and the algebra \((Q; \{+,-,\cdot,0,1\})\) satisfies the following identities:  
\[ x + y = x \cdot y, \quad \overline{\overline{x}} = x, \]

where \(\overline{x} = (\overline{x})(1, 2, 3, 4, 5, 6, 10, 12, 13, 15, 20, 21)\). For example, the standard fuzzy algebra \(F = (0, 1]; max(x, y), min(x, y), 1 - x, 0, 1\) is a De Morgan algebra.

For the definition of free algebras of the given variety see \([9, 22, 23]\).

Let \(B = \{0, 1\}\). Define the operations \(+, -, \cdot\), on \(B\) by the following way:  
\[ 0 + 0 = 0, \quad 0 + 1 = 1 + 0 = 1 + 1 = 1, \quad 0 - 1 = 0 \cdot 0 = 1 - 0 = 0, \quad 1 - 1 = 1, \quad 0 = 1, \quad 1 = 0. \]

We get the Boolean algebra \(2 = (B; \{+,-,\cdot,0,1\})\).

For a set \(X\) denote the set of all its subsets by \(2^X\) or \(\mathcal{P}(X)\). If we consider subsets of a given set \(X\) then for a subset \(s \subseteq X\) we denote \(\overline{s} = X \setminus s\).

A function \(f : B^n \rightarrow B\) is called a **Boolean function** of \(n\) variables. Let \(\leq\) be the order of the lattice \((B; \{+, \cdot\})\). For \(u = (u_1, \ldots, u_n)\), \(v = (v_1, \ldots, v_n) \in B^n\) we define:  
\[ u \leq v \text{ if and only if } u_i \leq v_i \text{ for all } i = 1, \ldots, n. \]

A Boolean function \(f : B^n \rightarrow B\) is called **monotone** if it preserves the order \(\leq\), i.e.,  
\[ x \leq y \Rightarrow f(x) \leq f(y), \]

where \(x, y \in B^n\).

**Definition 1.** An algebra \((Q; \{+,-,\cdot,0,1\})\) with two binary, two unary and two nullary operations is called a **Boole-De Morgan algebra** if \((Q; \{+,-,\cdot,0,1\})\) is a De Morgan algebra and \((Q; \{+,-,\cdot,0,1\})\) is a Boolean algebra and the two unary operations commute, i.e. \((\overline{x})' = (x')\).

This concept is introduced in \([14, 15]\) under the name of Boolean bisemigroup.

For example, every Boolean algebra can be considered as a Boole-De Morgan algebra with two equal unary operations. In particular if \(2 = (B; \{+,-,\cdot,0,1\})\) is the two-element Boolean algebra and \(\overline{\overline{x}} = x'\) (i.e. the unary operations \(-\) and \(\cdot\) are equal) then the algebra \((B; \{+,-,\cdot,0,1\})\) is a Boole-De Morgan algebra and we will denote it by \(BM_2\).

Further, denote \(D = \{0, a, b, 1\}\). Defining \(0 + x + 0 = x, \quad 0 \cdot x = x \cdot 0 = 0\) and \(1 + x = x = 1, x + x = 1\) for all \(x \in D\), and \(a + b = b + a = 1, ab = ba = 0, 0 = 1, \overline{\overline{a}} = a, a' = b, b' = a\) we get the De Morgan algebra \(4 = ((0, a, b, 1); \{+,-,\cdot,0,1\})\) and the Boole-De Morgan algebra \(BM_4 = (D; \{+,-,\cdot,0,1\})\).

For any Boolean algebra \(B\) and its dual Boolean algebra \(B^\text{op}\) consider the direct product \(B \times B^\text{op}\). Defining one more unary operation \(\ast\) by \((x,y) = (y,x)\) we get a Boole-De Morgan algebra.

For a Boole-De Morgan algebra \((Q; \{+,-,\cdot,\ast,0,1\})\) we define one more unary operation \(\ast\) in the following way: \(x^\ast = (x)^\ast = (x)\). It is easy to see that \((x + y)^\ast = x^\ast + y^\ast, (x \cdot y)^\ast = x^\ast \cdot y^\ast, \overline{x}^\ast = (\overline{x})\), \((x^\ast)^\ast = (x)^\ast\). Thus the mapping \(x \rightarrow x^\ast\) is an automorphism of the Boole-De Morgan algebra \((Q; \{+,-,\cdot,\ast,0,1\})\).

It is commonly known that the free Boolean algebra on \(n\) free generators is isomorphic to the Boolean algebra of Boolean functions of \(n\) variables \([3, 9, 22]\). The free bounded distributive lattice on \(n\) free generators is isomorphic to the lattice of monotone Boolean functions of...
In terms of clone theory Condition (1) means that the free De Morgan algebra on \( n \) free generators is isomorphic to the De Morgan algebra of De Morgan functions of \( n \) variables. This is a solution of the problem posed by B.I. Plotkin. In paper [21] we have also introduced the concept of quasi-De Morgan function and used them to establish some properties of De Morgan functions.

In this paper we continue the study of the properties of De Morgan and quasi-De Morgan functions and define disjunctive and conjunctive normal forms for those functions. And finally, we introduce the concept of Zhegalkin polynomial for the case of quasi-De Morgan functions and prove some results analogous to the Zhegalkin’s representation theorem on Boolean functions (see [7, 25]).

2. DE MORGAN AND QUASI-DE MORGAN FUNCTIONS

Recall that \( D = \{0, a, b, 1\} \) and \( B = \{0, 1\} \). Let us consider a one-to-one correspondence between the sets \( D \) and \( B \times B \) as follows:

\[
0 \leftrightarrow (0, 0), \ a \leftrightarrow (1, 0), \ b \leftrightarrow (0, 1), \ 1 \leftrightarrow (1, 1).
\]

We define the operations \( +, \cdot, ^{'} \) on the set \( B \times B \) as follows:

\[
(u, v) = (v', u'), \ (u, v) = (u', v'),
\]

\[
(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2),
\]

\[
(u_1, v_1) \cdot (u_2, v_2) = (u_1 \cdot u_2, v_1 \cdot v_2)
\]

(here the operations on the right hand side are the operations of the Boole-De Morgan algebra \( \text{BM}_2 \)). We get the Boole-De Morgan algebra \( (B \times B; \{+; \cdot; ^{'}\}, 0, 1) \), which is isomorphic to the algebra \( \text{BM}_4 \) (the one-to-one correspondence described above is an isomorphism).

However, if the tuple \( (y, z) \in B \times B \) corresponds to \( x \in D \) then we will write \( x = (y, z) \) (this causes no confusion).

For \( x \in D \) let

\[
x^* = \begin{cases} 
  x, & \text{if } x = 0, 1, \\
  a, & \text{if } x = b, \\
  b, & \text{if } x = a.
\end{cases}
\]

The unary operation \( ^* \) can also be defined on \( B \times B \) taking into account the isomorphism described above. In result we get \( (u, v)^* = (v', u') = (u^*, v^*) \), \( u, v \in B \). It is clear that \( x^* = (\overline{x})^* = \overline{x} \) (which agrees with the notation from the previous section).

**Definition 2.** A function \( f : D^n \rightarrow D \) is called a quasi-De Morgan function if the following conditions hold:

1. if \( x_i \in \{0, 1\}, \ i = 1, \ldots, n \), then \( f(x_1, \ldots, x_n) \in \{0, 1\} \);
2. if \( x_i \in D, \ i = 1, \ldots, n \), then \( f(x'_1, \ldots, x'_n) = (f(x_1, \ldots, x_n))^* \).

In terms of clone theory Condition (1) means that the function \( f \) preserves the unary relation \( \{0, 1\} \subseteq D \), and Condition (2) means that \( f \) preserves the binary relation \( \{\langle 0, 0 \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle 1, 1 \rangle\} \subseteq D^2 \), which is the graph of the automorphism \( x \mapsto x^* \) (for clones see [22, 16, 17]).

For \( c = (c_1, \ldots, c_n), \ d = (d_1, \ldots, d_n) \in D^n \) we say that \( d \) is a permitted modification of \( c \) if for some \( k (1 \leq k \leq n) \) we have \( d_i = c_i \) for all \( 1 \leq i \leq n, \ i \neq k \) and

\[
d_k = \begin{cases} 
  a, & \text{if } c_k = 0, \\
  1, & \text{if } c_k = b.
\end{cases}
\]

**Definition 3.** A quasi-De Morgan function \( f : D^n \rightarrow D \) is called a De Morgan function if it satisfies the following condition:

3. if \( x, y \in D^n \) with \( f(x) \neq b \) and \( y \) is a permitted modification of \( x \) then \( f(y) \in \{f(x), a\} \).

In terms of clone theory this condition means that \( f \) preserves the order relation \( \rho = \{\langle b, b \rangle, \langle b, 0 \rangle, \langle b, 1 \rangle, \langle b, a \rangle, \langle 0, 0 \rangle, \langle 0, a \rangle, \langle 1, 1 \rangle, \langle 1, a \rangle, \langle a, a \rangle\} \subseteq D^2 \).

Notice that Condition (1) is a consequence of Condition (2), but, however, it is more convenient to write it as a separate condition.

Note that it follows from Condition (1) that every quasi-De Morgan function is an extension of some Boolean function. And notice that the constant functions \( f = 1 \) and \( f = 0 \) are quasi-De Morgan functions, but the constant functions \( f = a \) and \( f = b \) are not. This means that \( 0 \) and \( 1 \) are the only constant quasi-De Morgan functions. Further examples of quasi-De Morgan functions are \( f(x) = x, \ g(x) = \overline{x}, \ h(x, y) = x + y, \ q(x, y) = x + y, \ p(x) = x^* \), where the operations on the right hand side are the operations of the Boole-De Morgan algebra \( \text{BM}_4 \). Also note that the function \( p \) is an example of quasi-De Morgan function which is not a De Morgan function.

As Boolean and De Morgan functions (and also all functions \( D^n \rightarrow D \)), quasi-De Morgan functions can be given by tables. Also note that there is an algorithm which for a given table of a function \( f : D^n \rightarrow D \) determines whether \( f \) is a quasi-De Morgan function.

Below, for \( x_i \in D \) we denote by \( (y_i, z_i) \) the pair from \( B \times B \) which corresponds to \( x_i \), i.e. \( x_i = (y_i, z_i) \).

**Theorem 1.** A function \( f : D^n \rightarrow D \) is a quasi-De Morgan function if and only if there exists a Boolean function \( \varphi : \mathbb{B}^{2n} \rightarrow B \) such that

\[
f(x_1, \ldots, x_n) = (\varphi(y_1, \ldots, y_n, z'_1, \ldots, z'_n), \varphi(z_1, \ldots, z_n, y'_1, \ldots, y'_n)),
\]

for all \( x_1, \ldots, x_n \in D \).

For a quasi-De Morgan function \( f : D^n \rightarrow D \) there exists a unique Boolean function \( \varphi : \mathbb{B}^{2n} \rightarrow B \) which satisfies (1). To emphasize that \( \varphi \) is the unique Boolean function corresponding to \( f \), we denote it by \( \varphi_f \).

**Corollary 1.** There are exactly \( 2^{4n} \) quasi-De Morgan functions of \( n \) variables.

Let us formulate the following criterion.

**Theorem 2.** A quasi-De Morgan function \( f : D^n \rightarrow D \) is a De Morgan function if and only if the corresponding Boolean function \( \varphi_f \) is monotone.
Let $m_n$ be the number of monotone Boolean functions of $n$ variables (those numbers are called Dedekind numbers too).

**Corollary 2.** There are $m_{2n}$ De Morgan functions of $n$ variables.

Denote the set of all quasi-De Morgan functions of $n$ variables by $BM_n$. For the functions $f, g : D^n \to D$ define $f + g$, $f \cdot g$, $\overline{f}$ and $f'$ in the standard way, i.e. $(f + g)(x) = f(x) + g(x)$, $(f \cdot g)(x) = f(x) \cdot g(x)$, $f(x) = \overline{f(x)}$, $x \in D^n$, where the operations on the right hand side are the operations of the Boole-De Morgan algebra $BM_4$. The set $BM_n$ is closed under operations $+$, $\cdot$ and so we get an algebra: $BM_n = (BM_n, \{+\cdot, 0, 1\})$ (here 0 and 1 are the constant quasi-De Morgan functions), which obviously is a Boole-De Morgan algebra. The analogous result is valid for De Morgan functions too, i.e. the set of all De Morgan functions of $n$ variables is closed under operations $+$, $\cdot$.

And so if $D_n$ is the set of all De Morgan functions of $n$ variables then $D_n = (D_n, \{+\cdot, 0, 1\})$ is a De Morgan algebra.

For a set $S \subseteq 2^{\{1, \ldots, n\}} \times 2^{\{1, \ldots, n\}}$ define the function $f_S : D^n \to D$ in the following way:

$$f_S(x_1, \ldots, x_n) = \sum_{(s_1, s_2) \in S} \left( \prod_{i \in s_1} x_i \cdot \prod_{i \in s_2} \overline{x_i} \right),$$

(2)

where the operations on the right hand side are the operations of $BM_n$. Notice that $f_S$ does not depend on the order of the elements in the set $S$.

Also we set $f_{\emptyset} = 0$ and $f_{\{(0, 0)\}} = 1$.

Let us consider the projection functions

$$\delta^i_i = x_i, \quad i = 1, \ldots, n,$$

as functions $\delta^D \to D$. Obviously, $\delta^i$ is a quasi-De Morgan function for each $i$. And according to (2), for any set $S \subseteq 2^{\{1, \ldots, n\}} \times 2^{\{1, \ldots, n\}}$ we have:

$$f_S = \sum_{(s_1, s_2) \in S} \left( \prod_{i \in s_1} \delta^i_i \cdot \prod_{i \in s_2} \overline{\delta^i_i} \right).$$

Hence, $f_S \in BM_n$, i.e. $f_S$ is a quasi-De Morgan function for any set $S \subseteq 2^{\{1, \ldots, n\}} \times 2^{\{1, \ldots, n\}}$.

For $s = (s_1, s_2) \in 2^{\{1, \ldots, n\}} \times 2^{\{1, \ldots, n\}}$ let $s' = s_1 \cup \{n + i : i \in s_2\} \subseteq 2^{\{1, \ldots, 2n\}}$, and for $S' = \{s' : s \in S\} \subseteq 2^{\{1, \ldots, 2n\}}$. In this way we give a one-to-one correspondence between the sets $P \left( 2^{\{1, \ldots, n\}} \times 2^{\{1, \ldots, n\}} \right)$ and $P \left( 2^{\{1, \ldots, 2n\}} \right)$.

Now, for any quasi-De Morgan function $f \in BM_n$ from Theorem 1 we conclude that there exists a set $S' \subseteq 2^{\{1, \ldots, 2n\}}$ such that:

$$f(x_1, \ldots, x_n) = \sum_{(s_1, s_2) \in S'} \left( \prod_{i \in s_1} x_i \cdot \prod_{i \in s_2} \overline{x_i} \right),$$

where $S$ is the subset of $2^{\{1, \ldots, n\}} \times 2^{\{1, \ldots, n\}}$ corresponding to $S'$. Moreover, the number of all quasi-De Morgan functions of $n$ variables is the same as the number of all subsets of $2^{\{1, \ldots, n\}} \times 2^{\{1, \ldots, n\}}$. Hence, we get the following result.

**Theorem 3.** For any quasi-De Morgan function $f$ of $n$ variables there exists a unique set $S \subseteq 2^{\{1, \ldots, n\}} \times 2^{\{1, \ldots, n\}}$ such that $f = f_S$.

In particular, $f_S \neq f_{S'}$ if $S \neq S'$. Thus, every quasi-De Morgan function can be uniquely presented in the form (2). This form is called the disjunctive normal form (or briefly - DNF) of quasi-De Morgan function $f$.

Below we will use the concept of essential variable (and essential dependence) of quasi-De Morgan functions. The definitions are the same as in case of Boolean functions and so we do not give them here.

**Theorem 4.** For any quasi-De Morgan function $f$ the corresponding Boolean function $\varphi_f$ does not essentially depend on the last $n$ variables if and only if $f$ can be represented as a term function with functional symbols $\cdot, +$, i.e. $f$ is a term function of the Boolean algebra $(D; \{+, \cdot, 0, 1\})$.

**Proof.** Notice that in the representation of a quasi-De Morgan function $f$ by DNF the terms of the form $x_i$ or $x'_i$ correspond to the first $n$ variables of the Boolean function $\varphi_f$, and the terms of the form $x_i$ and $x'_i$ correspond to the last $n$ variables of $\varphi_f$. Hence, if $\varphi_f$ does not essentially depend on the last $n$ variables then $f$ can be represented in the following form for some set $S \subseteq 2^{\{1, \ldots, n\}}$:

$$f(x_1, \ldots, x_n) = \sum_{s \in S} \left( \prod_{i \in s} x_i \cdot \prod_{i \notin s} \overline{x_i} \right).$$

(3)

Obviously there are $2^m$ such terms and they are different (as functions) for different sets $S$. Also there are $2^m$ quasi-De Morgan functions for which the corresponding Boolean function $\varphi_f$ does not essentially depend on the last $n$ variables. Therefore only those functions for which the corresponding Boolean function does not essentially depend on the last $n$ variables can be represented in the form (3). And also the representation is unique. And finally note that all terms of the Boolean algebra $(D; \{+, \cdot, 0, 1\})$ (and also of any Boolean algebra) can be reduced to the form (3) (it is well known). \[ \square \]

By similar arguments we can establish the analogous representations for De Morgan functions. To formulate that result let us recall the definition of a antichain. If $(L; \leq)$ is a partially ordered set then $A \subseteq L$ is called an antichain if any two different elements of the set $A$ are incomparable (for antichain see [9, 8, 11, 24]). Defining the order relation $\subseteq$ on the set $2^{\{1, \ldots, n\}} \times 2^{\{1, \ldots, n\}}$ componentwise (i.e. $(a_1, a_2) \subseteq (b_1, b_2)$, if $a_1 \subseteq b_1$ and $a_2 \subseteq b_2$) we get a partially ordered set $2^{\{1, \ldots, n\}} \times 2^{\{1, \ldots, n\}}$. Below we will use antichains of that poset.

**Theorem 5.** Every De Morgan function can be uniquely represented in the following form:

$$f(x_1, \ldots, x_n) = \sum_{s \in S} \left( \prod_{i \in s} x_i \cdot \prod_{i \notin s} \overline{x_i} \right),$$

where $S \subseteq 2^{\{1, \ldots, n\}} \times 2^{\{1, \ldots, n\}}$ is an antichain.
This form is called the disjunctive normal form for De Morgan function $f$.

### 3. REPRESENTATION BY POLYNOMIALS

First recall that in the theory of Boolean functions the exclusive-or function is defined as follows:

$$ x \oplus y = xy' + x'y. $$

This is a binary operation on the set $B$ and the algebra $(B; \{\oplus, \cdot\})$ is a field (it is the two-element Galois field $\text{GF}(2) = \mathbb{Z}_2$).

Recall the following well known result.

**Theorem 6.** ([7]) For every Boolean function $f : B^n \to B$ there exists a unique mapping $\alpha : \mathcal{P}(\{1, \ldots, n\}) \to \{0, 1\}$ such that

$$ f(x_1, \ldots, x_n) = \bigoplus_{A \subseteq \mathcal{P}(\{1, \ldots, n\})} \alpha(A) \prod_{i \in A} x_i, \quad (4) $$

where the operations on the right hand side are the operations of the two-element field $\text{GF}(2) = \mathbb{Z}_2$.

The right hand side of (4) is a polynomial modulo 2. Those polynomials are called also Zhegalkin polynomials for Boolean functions [25].

Let us recall the one-to-one correspondence between the sets $D$ and $B \times B$ described:

$$ 0 \leftrightarrow (0, 0), \ a \leftrightarrow (1, 0), \ b \leftrightarrow (0, 1), \ 1 \leftrightarrow (1, 1). $$

Now we can define an operation $\oplus$ on $B \times B$ componentwisely (i.e. $(u_1, v_1) \oplus (u_2, v_2) = (u_1 \oplus u_2, v_1 \oplus v_2)$) and taking into account the above one-to-one correspondence we can define the operation $\oplus$ on $D$. We get the following table of that operation:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>a</th>
<th>b</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>a</td>
<td>b</td>
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<tr>
<td>1</td>
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<td>b</td>
<td>a</td>
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<td>a</td>
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<td>b</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The algebras $(D; \{\oplus, \cdot\})$ and $(B \times B; \{\oplus, \cdot\})$ are isomorphic. Also notice that $x \oplus y = xy' + x'y'$, where the operations on the right hand side are the operations of the Boole-De Morgan algebra $\text{BM}_4$.

Now we can define Zhegalkin polynomials on the set $D$. Those are the polynomials of the form (4), only the domain of variables now is the four-element set $D$, and $\oplus$ is the operation on the set $D$ defined above and $\cdot$ is the multiplication of the Boole-De Morgan algebra $\text{BM}_4$. The functions $x \oplus y$, $x \cdot y$ and constant functions 0, 1 are quasi-De Morgan functions, hence all Zhegalkin polynomials on the set $D$ considered as functions from $D^n$ into $D$ are quasi-De Morgan functions. Here we restrict the coefficients to be only 0 and 1 (and not $a$ or $b$) because the constant functions $a$ and $b$ are not quasi-De Morgan functions and so if we let the coefficients to be $a$ or $b$ then the polynomial could be a non quasi-De Morgan function. Two Zhegalkin polynomials on the set $D$ are called equal if the corresponding coefficients are equal.

**Theorem 7.** A quasi-De Morgan function $f : D^n \to D$ can be represented as a Zhegalkin polynomial on the set $D$ if and only if the corresponding Boolean function $\varphi_I(t_1, \ldots, t_{2n})$ does not essentially depend on the variables $t_{n+1}, \ldots, t_{2n}$. And if this is the case then the representation is unique.

This theorem shows that two Zhegalkin polynomials on the set $D$ are equal (as polynomials) if and only if they are equal as functions, i.e. their values are equal for any values of variables.

It is easy to verify that for a quasi-De Morgan function $f$ the corresponding Boolean function $\varphi_I$ does not essentially depend on the last $n$ variables if and only if $f$ preserves the binary relation $\sigma = \{(0, 0), (0, b), (b, 0), (0, b), (a, a), (a, 1), (1, 1), (1, a)\} \subseteq D^2$. Thus, a quasi-De Morgan function is a Zhegalkin polynomial on the set $D$ if and only if it preserves the binary relation $\sigma$.

From Theorem 4 we deduce that the term functions of the four-element Boolean algebra $(D; \{+,-, \cdot, 0, 1\})$ and only they can be represented as Zhegalkin polynomials on the set $D$. This result also follows from the following equalities: $u' = 1 \oplus u$, $u \in D$ and $u + v = (u' \cdot v') = 1 \oplus (1 \oplus u) \cdot (1 \oplus v) = u \oplus v \oplus u \cdot v$, $u, v \in D$. And these equalities give us an algorithm which, given a quasi-De Morgan function, gives its representation as a Zhegalkin polynomial on the set $D$.

A quasi-De Morgan function $f$ is called monotone if it preserves the order of the lattice $(D; \{+,-, \cdot\})$.

It is easy to see that if $f$ and $g$ are monotone De Morgan functions then $f \cdot g$ and $f + g$ are monotone, too. Hence, all De Morgan functions of the following form are monotone De Morgan functions:

$$ \sum \prod_{x_i \in S} x_i, \quad (5) $$

where $S \subseteq 2^{\{1, \ldots, n\}}$ is an antichain. In the proof of the following theorem we also show that all monotone De Morgan functions can be presented in the form (5).

**Theorem 8.** A De Morgan function $f(x_1, \ldots, x_n)$ is monotone if and only if the corresponding Boolean function $\varphi_I(t_1, \ldots, t_{2n})$ does not essentially depend on the variables $t_{n+1}, \ldots, t_{2n}$.

Now we deduce the following result.

**Corollary 3.** A De Morgan function can be represented as a Zhegalkin polynomial on the set $D$ if and only if it is monotone.

The idea of De Morgan and quasi-De Morgan functions is also applicable in quantum computation, quantum information theory, quantum logic and the theory of quantum computers.
REFERENCES


