Computation of De Morgan and Quasi-De Morgan Functions

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ABSTRACT

In this paper we give various algorithms for computation of De Morgan and quasi-De Morgan functions. We define the disjunctive and conjunctive normal forms for De Morgan and quasi-De Morgan functions and introduce the concept of Zhegalkin polynomial for the case of quasi-De Morgan functions as well as prove some results analogous to the Zhegalkin representation theorem on Boolean functions.

Keywords

Antichain, Boolean function, monotone Boolean function, De Morgan and quasi-De Morgan function, disjunctive (conjunctive) normal form, Zhegalkin-type representation.

1. INTRODUCTION AND PRELIMINARIES

An algebra $(Q;\{+,\cdot,\ ',0,1\})$ with two binary, one unary and two nullary operations is called a *Boolean algebra* if $(Q;\{+,\cdot,0,1\})$ is a bounded distributive lattice with least element 0 and greatest element 1 and the algebra $(Q;\{+,\cdot,\ ',0,1\})$ satisfies the following identities:

$$x+x'=1, \quad x\cdot x'=0.$$

An algebra $(Q;\{+,\cdot,^{-},0,1\})$ with two binary, one unary and two nullary operations is called a *De Morgan algebra* if $(Q;\{+,\cdot,0,1\})$ is a bounded distributive lattice with least element 0 and greatest element 1 and the algebra $(Q;\{+,\cdot,^{-},\})$ satisfies the following identities:

$$\overline{x+y}=\overline{x}\cdot\overline{y}, \quad \overline{\overline{x}}=x,$$

where $\overline{\overline{x}}=\overline{(\overline{x})}$([1, 2, 3, 4, 5, 6, 10, 12, 13, 15, 20, 21]). For example, the standard fuzzy algebra $F=([0,1];max(x,y),min(x,y),1-x,0,1)$ is a De Morgan algebra.

For the definition of free algebras of the given variety see [9, 22, 23].

Let $B=\{0,1\}$. Define the operations $+,\cdot,\ '$, on B by the following way: $0+0=0,\ 0+1=1+0=1+1=1,\ 0\cdot 1=0\cdot 0=1\cdot 0=0,\ 1\cdot 1=1,\ 0'=1,\ 1'=0$. We get the Boolean algebra $\mathbf{2}=(B;\{+,\cdot,\ ',0,1\})$.

For a set X denote the set of all its subsets by 2^X or $\mathcal{P}(X)$. If we consider subsets of a given set X then for a subset $s\subseteq X$ we denote $\overline{s}=X\setminus s$.

A function $f:B^n\to B$ is called a *Boolean function* of n variables. Let "$\leq$" be the order of the lattice $(B;\{+,\cdot\})$. For $u=(u_1,\ldots,u_n),\ v=(v_1,\ldots,v_n)\in B^n$ we define: $u\leq v$ if and only if $u_i\leq v_i$ for all $i=1,\ldots,n$. A Boolean function $f:B^n\to B$ is called *monotone* if it preserves the order "$\leq$", i.e.

$$x\leq y\Rightarrow f(x)\leq f(y),$$

where $x,y\in B^n$.

Definition 1. *An algebra $(Q;\{+,\cdot,^{-},',0,1\})$ with two binary, two unary and two nullary operations is called a Boole-De Morgan algebra if $(Q;\{+,\cdot,^{-},0,1\})$ is a De Morgan algebra and $(Q;\{+,\cdot,\ ',0,1\})$ is a Boolean algebra and the two unary operations commute, i.e. $(\overline{x})'=\overline{(x')}$.*

This concept is introduced in [14, 15] under the name of Boolean bisemigroup.

For example, every Boolean algebra can be considered as a Boole-De Morgan algebra with two equal unary operations. In particular if $\mathbf{2}=(B;\{+,\cdot,\ ',0,1\})$ is the two-element Boolean algebra and $\overline{x}=x'$ (i.e. the unary operations $^{-}$ and $'$ are equal) then the algebra $(B;\{+,\cdot,^{-},',0,1\})$ is a Boole-De Morgan algebra and we will denote it by $\mathcal{BM}_2$.

Further, denote $D=\{0,a,b,1\}$. Defining $0+x=x+0=x,\ 0\cdot x=x\cdot 0=0$ and $1\cdot x=x\cdot 1=x,\ 1+x=x+1=1$ for all $x\in D$, and $a+b=b+a=1,\ ab=ba=0,\ \overline{0}=1,\ \overline{1}=0,\ \overline{a}=a,\ \overline{b}=b,\ 1'=0,\ 0'=1,\ a'=b,\ b'=a$ we get the De Morgan algebra $\mathbf{4}=(\{0,a,b,1\};\{+,\cdot,^{-},0,1\})$ and the Boole-De Morgan algebra $\mathcal{BM}_4=(D;\{+,\cdot,^{-},\ ',0,1\})$.

For any Boolean algebra $\mathcal{B}$ and its dual Boolean algebra B^{op} consider the direct product $\mathcal{B}\times\mathcal{B}^{op}$. Defining one more unary operation $^{-}$ by $\overline{(x,y)}=(y,x)$ we get a Boole-De Morgan algebra.

For a Boole-De Morgan algebra $(Q;\{+,\cdot,^{-},',0,1\})$ we define one more unary operation * in the following way: $x^*=\overline{(x)'}=(\overline{x'})$. It is easy to see that $(x+y)^*=x^*+y^*,\ (x\cdot y)^*=x^*\cdot y^*,\ \overline{x}^*=\overline{(x)^*},\ (x^*)'=(x')^*$. Thus the mapping $x\to x^*$ is an automorphism of the Boole-De Morgan algebra $(Q;\{+,\cdot,^{-},',0,1\})$.

It is commonly known that the free Boolean algebra on n free generators is isomorphic to the Boolean algebra of Boolean functions of n variables [3, 9, 22]. The free bounded distributive lattice on n free generators is isomorphic to the lattice of monotone Boolean functions of

n variables. In paper [21] we have introduced the concept of De Morgan function and proved that the free De Morgan algebra on n free generators is isomorphic to the De Morgan algebra of De Morgan functions of n variables. This is a solution of the problem posed by B.I. Plotkin. In paper [21] we have also introduced the concept of quasi-De Morgan functions and used them to establish some properties of De Morgan functions.

In this paper we continue the study of the properties of De Morgan and quasi-De Morgan functions and define disjunctive and conjunctive normal forms for those functions. And finally, we introduce the concept of Zhegalkin polynomial for the case of quasi-De Morgan functions and prove some results analogous to the Zhegalkin's representation theorem on Boolean functions (see [7, 25]).

2. DE MORGAN AND QUASI-DE MORGAN FUNCTIONS

Recall that $D = \{0, a, b, 1\}$ and $B = \{0, 1\}$. Let us consider a one-to-one correspondence between the sets D and $B \times B$ as follows:

$$0 \leftrightarrow (0, 0),\ a \leftrightarrow (1, 0),\ b \leftrightarrow (0, 1),\ 1 \leftrightarrow (1, 1).$$

We define the operations $+, \cdot, ^{-}, '$ on the set $B \times B$ as follows:

$$\begin{array}{l} \overline{(u, v)} = (v', u'),\ (u, v)' = (u', v'), \\ (u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2), \\ (u_1, v_1) \cdot (u_2, v_2) = (u_1 \cdot u_2, v_1 \cdot v_2) \end{array}$$

(here the operations on the right hand side are the operations of the Boole-De Morgan algebra $\mathcal{BM}_2$). We get the Boole-De Morgan algebra $(B \times B; \{+, \cdot, ^{-}, ', 0, 1\})$, which is isomorphic to the algebra $\mathcal{BM}_4$ (the one-to-one correspondence described above is an isomorphism). However, if the tuple $(y, z) \in B \times B$ corresponds to $x \in D$ then we will write $x = (y, z)$ (this causes no confusion).

For $x \in D$ let

$$x^* = \begin{cases} x, & \text{if } x = 0, 1, \\ a, & \text{if } x = b, \\ b, & \text{if } x = a. \end{cases}$$

The unary operation * can also be defined on $B \times B$ taking into account the isomorphism described above. In result we get $(u, v)^* = \overline{(v, u)} = (v^*, u^*),\ u, v \in B$. It is clear that $x^* = (\overline{x})' = \overline{x'}$ (which agrees with the notation from the previous section).

Definition 2. *A function $f : D^n \to D$ is called a quasi-De Morgan function if the following conditions hold:*
(1) if $x_i \in \{0, 1\},\ i = 1, \dots, n$, then $f(x_1, \dots, x_n) \in \{0, 1\}$,
(2) if $x_i \in D,\ i = 1, \dots, n$, then $f(x_1^*, \dots, x_n^*) = (f(x_1, \dots, x_n))^*$,

In terms of clone theory Condition (1) means that the function f preserves the unary relation $\{0, 1\} \subseteq D$, and Condition (2) means that f preserves the binary relation $\{(0, 0), (a, b), (b, a), (1, 1)\} \subseteq D^2$, which is the graph of the automorphism $x \mapsto x^*$ (for clones see [22, 16, 17]).

For $c = (c_1, \dots, c_n), d = (d_1, \dots, d_n) \in D^n$ we say that d is a *permitted modification* of c if for some k $(1 \leq k \leq n)$ we have $d_i = c_i$ for all $1 \leq i \leq n,\ i \neq k$ and

$$d_k = \begin{cases} a, & \text{if } c_k = 0, \\ 1, & \text{if } c_k = b. \end{cases}$$

Definition 3. *A quasi-De Morgan function $f : D^n \to D$ is called a De Morgan function if it satisfies the following condition:*
(3) if $x, y \in D^n$ with $f(x) \neq b$ and y is a permitted modification of x then $f(y) \in \{f(x), a\}$.

In terms of clone theory this condition means that f preserves the order relation $\rho = \{(b, b), (b, 0), (b, 1), (b, a), (0, 0), (0, a), (1, 1), (1, a), (a, a)\} \subseteq D^2$.

Notice that Condition (1) is a consequence of Condition (2), but, however, it is more convenient to write it as a separate condition.

Note that it follows from Condition (1) that every quasi-De Morgan function is an extension of some Boolean function. And notice that the constant functions $f = 1$ and $f = 0$ are quasi-De Morgan functions, but the constant functions $f = a$ and $f = b$ are not. This means that 0 and 1 are the only constant quasi-De Morgan functions. Further examples of quasi-De Morgan functions are $f(x) = x,\ g(x) = \overline{x},\ h(x, y) = x \cdot y,\ q(x, y) = x + y,\ p(x) = x'$, where the operations on the right hand side are the operations of the Boole-De Morgan algebra $\mathcal{BM}_4$. Also note that the function p is an example of quasi-De Morgan function which is not a De Morgan function.

As Boolean and De Morgan functions (and also all functions $D^n \to D$), quasi-De Morgan functions can be given by tables. Also note that there is an algorithm which for a given table of a function $f : D^n \to D$ determines whether f is a quasi-De Morgan function.

Below, for $x_i \in D$ we denote by (y_i, z_i) the pair from $B \times B$ which corresponds to x_i, i.e. $x_i = (y_i, z_i)$.

Theorem 1. *A function $f : D^n \to D$ is a quasi-De Morgan function if and only if there exists a Boolean function $\varphi : B^{2n} \to B$ such that*

$$f(x_1, \dots, x_n) = (\varphi(y_1, \dots, y_n, z_1', \dots, z_n'), \varphi(z_1, \dots, z_n, y_1', \dots, y_n')), \tag{1}$$

for all $x_1, \dots, x_n \in D$.

For a quasi-De Morgan function $f : D^n \to D$ there exists a unique Boolean function $\varphi : B^{2n} \to B$ which satisfies (1). To emphasize that φ is the unique Boolean function corresponding to f, we denote it by φ_f.

Corollary 1. *There are exactly 2^{4^n} quasi-De Morgan functions of n variables.*

Let us formulate the following criterion.

Theorem 2. *A quasi-De Morgan function $f : D^n \to D$ is a De Morgan function if and only if the corresponding Boolean function φ_f is monotone.*

Let m_n be the number of monotone Boolean functions of n variables (those numbers are called Dedekind numbers too).

Corollary 2. *There are m_{2n} De Morgan functions of n variables.*

Denote the set of all quasi-De Morgan functions of n variables by $\mathcal{BM}_n$. For the functions $f, g : D^n \to D$ define $f+g$, $f\cdot g$, $\overline{f}$ and f' in the standard way, i.e. $(f \pm g)(x) = f(x)+g(x)$, $(f\cdot g)(x) = f(x)\cdot g(x)$, $\overline{f}(x) = \overline{f(x)}$, $f'(x) = (f(x))'$, $x \in D^n$, where the operations on the right hand side are the operations of the Boole-De Morgan algebra $\mathcal{BM}_4$. The set $\mathcal{BM}_n$ is closed under operations $+, \cdot, ^{-}, '$ and so we get an algebra: $BM_n = (\mathcal{BM}_n, \{+, \cdot, ^{-}, ', 0, 1\})$ (here 0 and 1 are the constant quasi-De Morgan functions), which obviously is a Boole-De Morgan algebra. The analogous result is valid for De Morgan functions too, i.e. the set of all De Morgan functions of n variables is closed under operations $+, \cdot, ^{-}$. And so if $\mathcal{D}_n$ is the set of all De Morgan functions of n variables then $D_n = (\mathcal{D}_n, \{+, \cdot, ^{-}, 0, 1\})$ is a De Morgan algebra.

For a set $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ define the function $f_S : D^n \to D$ in the following way:

$$f_S(x_1,\ldots,x_n) = \sum_{(s_1,s_2)\in S} \left(\prod_{i\in s_1} x_i \cdot \prod_{i\in \overline{s}_1} x_i' \cdot \prod_{i\in s_2} \overline{x}_i \cdot \prod_{i\in \overline{s}_2} x_i^*\right), \tag{2}$$

where the operations on the right hand side are the operations of $\mathcal{BM}_4$. Notice that f_S does not depend on the order of the elements in the set S.

Also we set $f_\varnothing = 0$ and $f_{\{(\varnothing,\varnothing)\}} = 1$.

Let us consider the projection functions

$$\delta_n^i = x_i,\ i = 1,\ldots,n,$$

as functions $D^n \to D$. Obviously, δ_n^i is a quasi-De Morgan function for each i. And according to (2), for any set $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ we have:

$$f_S = \sum_{(s_1,s_2)\in S} \left(\prod_{i\in s_1} \delta_n^i \cdot \prod_{i\in \overline{s}_1} (\delta_n^i)' \cdot \prod_{i\in s_2} \overline{\delta_n^i} \cdot \prod_{i\in \overline{s}_2} (\delta_n^i)^*\right).$$

Hence, $f_S \in \mathcal{BM}_n$, i.e. f_S is a quasi-De Morgan function for any set $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$.

For $s = (s_1, s_2) \in 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ let $s' = s_1 \cup \{n+i : i \in s_2\} \in 2^{\{1,\ldots,2n\}}$, and for $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ let $S' = \{s' : s \in S\} \subseteq 2^{\{1,\ldots,2n\}}$. In this way we give a one-to-one correspondence between the sets $\mathcal{P}\left(2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}\right)$ and $\mathcal{P}\left(2^{\{1,\ldots,2n\}}\right)$.

Now, for any quasi-De Morgan function $f \in \mathcal{BM}_n$ from Theorem 1 we conclude that there exists a set $S' \subseteq 2^{\{1,\ldots,2n\}}$ such that:

$$\begin{gathered} f(x_1,\ldots,x_n) = \\ (\varphi_f(y_1,\ldots,y_n,z_1',\ldots,z_n'), \varphi_f(z_1,\ldots,z_n,y_1',\ldots,y_n')) = \\ \sum_{(s_1,s_2)\in S} \left(\prod_{i\in s_1} x_i \cdot \prod_{i\in \overline{s}_1} x_i' \cdot \prod_{i\in s_2} \overline{x}_i \cdot \prod_{i\in \overline{s}_2} x_i^*\right) = \\ f_S(x_1,\ldots,x_n), \end{gathered}$$

where S is the subset of $2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ corresponding to S'.
Moreover, the number of all quasi-De Morgan functions of n variables is the same as the number of all subsets of $2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$. Hence, we get the following result.

Theorem 3. *For any quasi-De Morgan function f of n variables there exists a unique set $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ such that $f = f_S$.*

In particular, $f_{S_1} \neq f_{S_2}$ if $S_1 \neq S_2$ (cf. [18, 19, 20]).

Thus, every quasi-De Morgan function can be uniquely presented in the form (2). This form is called the dis-junctive normal form (or briefly - DNF) of quasi-De Morgan function f.

Below we will use the concept of essential variable (and essential dependence) of quasi-De Morgan functions. The definitions are the same as in case of Boolean functions and so we do not give them here.

Theorem 4. *For a quasi-De Morgan function f the corresponding Boolean function φ_f does not essentially depend on the last n variables if and only if f can be represented as a term function with functional symbols $+, \cdot, '$, i.e. f is a term function of the Boolean algebra $(D; \{+, \cdot, ', 0, 1\})$.*

Proof. Notice that in the representation of a quasi-De Morgan function f by DNF the terms of the form x_i or x_i' correspond to the first n variables of the Boolean function φ_f, and the terms of the form $\overline{x}_i$ and x_i^* correspond to the last n variables of φ_f. Hence, if φ_f does not essentially depend on the last n variables then f can be represented in the following form for some set $S \subseteq 2^{\{1,\ldots,n\}}$:

$$f(x_1,\ldots,x_n) = \sum_{s\in S} \left(\prod_{i\in s} x_i \cdot \prod_{i\in \overline{s}} x_i'\right). \tag{3}$$

Obviously there are 2^{2^n} such terms and they are different (as functions) for different sets S_1 and S_2. Also there are 2^{2^n} quasi-De Morgan functions for which the corresponding Boolean function φ_f does not essentially depend on the last n variables. Therefore only those functions for which the corresponding Boolean function does not essentially depend on the last n variables can be represented in the form (3). And also the representation is unique. And finally note that all terms of the Boolean algebra $(D; \{+, \cdot, ', 0, 1\})$ (and also of any Boolean algebra) can be reduced to the form (3) (it is well known). $\square$

By similar arguments we can establish the analogous representations for De Morgan functions. To formulate that result let us recall the definition o f a ntichain. If $(L; \leq)$ is a partially ordered set then $A \subseteq L$ is called an *antichain* if any two different elements of the set A are incomparable (for antichain see [9, 8, 11, 24]). Defining the order relation $\subseteq$ on the set $2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ componentwisely (i.e. $(a_1, a_2) \subseteq (b_1, b_2)$, if $a_1 \subseteq b_1$ and $a_2 \subseteq b_2$) we get a partially ordered set $2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}(\subseteq)$. Below we will use antichains of that poset.

Theorem 5. *Every De Morgan function can be uniquely represented in the following form:*

$$f(x_1,\ldots,x_n) = \sum_{s=(s_1,s_2)\in S} \left(\prod_{i\in s_1} x_i \cdot \prod_{i\in s_2} \overline{x}_i\right),$$

where $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ is an antichain.

This form is called the disjunctive normal form for De Morgan function f.

3. REPRESENTATION BY POLYNOMIALS

First recall that in the theory of Boolean functions the *exclusive-or* function is defined as follows:

$$x \oplus y = xy' + x'y.$$

This is a binary operation on the set B and the algebra $(B; \{\oplus, \cdot\})$ is a field (it is the two-element Galois field $\mathbf{GF(2)} = Z_2$).

Recall the following well known result.

Theorem 6. ([7]) *For every Boolean function $f : B^n \to B$ there exists a unique mapping $\alpha : \mathcal{P}(\{1,\ldots,n\}) \to \{0,1\}$ such that*

$$f(x_1,\ldots,x_n) = \bigoplus_{A\in\mathcal{P}(\{1,\ldots,n\})} \alpha(A) \cdot \prod_{i\in A} x_i, \qquad (4)$$

where the operations on the right hand side are the operations of the two-element field $\mathbf{GF(2)} = Z_2$.

The right hand side of (4) is a polynomial modulo 2. Those polynomials are called also Zhegalkin polynomials for Boolean functions [25].

Let us recall the one-to-one correspondence between the sets D and $B \times B$ described:

$$0 \leftrightarrow (0,0),\ a \leftrightarrow (1,0),\ b \leftrightarrow (0,1),\ 1 \leftrightarrow (1,1).$$

Now we can define an operation $\oplus$ on $B \times B$ componentwisely (i.e. $(u_1, v_1) \oplus (u_2, v_2) = (u_1 \oplus u_2, v_1 \oplus v_2)$) and taking into account the above one-to-one correspondence we can define the operation $\oplus$ on D. We get the following table of that operation:

$\oplus$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

The algebras $(D; \{\oplus, \cdot\})$ and $(B\times B; \{\oplus, \cdot\})$ are isomorphic. Also notice that $x \oplus y = x'y + xy'$, where the operations on the right hand side are the operations of the Boole-De Morgan algebra $\mathcal{BM}_4$.

Now we can define *Zhegalkin polynomials* on the set D. Those are the polynomials of the form (4), only the domain of variables now is the four-element set D, and $\oplus$ is the operation on the set D defined above and $\cdot$ is the multiplication of the Boole-De Morgan algebra $\mathcal{BM}_4$. The functions $x \oplus y$, $x \cdot y$ and constant functions 0, 1 are quasi-De Morgan functions, hence all Zhegalkin polynomials on the set D considered as functions from D^n into D are quasi-De Morgan functions. Here we restrict the coefficients to be only 0 and 1 (and not a or b) because the constant functions a and b are not quasi-De Morgan functions and so if we let the coefficients to be a or b then the polynomial could be a non quasi-De Morgan function. Two Zhegalkin polynomials on the set D are called equal if the corresponding coefficients are equal.

Theorem 7. *A quasi-De Morgan function $f : D^n \to D$ can be represented as a Zhegalkin polynomial on the set D if and only if the corresponding Boolean function $\varphi_f(t_1,\ldots,t_{2n})$ does not essentially depend on the variables $t_{n+1},\ldots,t_{2n}$. And if this is the case then the representation is unique.*

This theorem shows that two Zhegalkin polynomials on the set D are equal (as polynomials) if and only if they are equal as functions, i.e. their values are equal for any values of variables.

It is easy to verify that for a quasi-De Morgan function f the corresponding Boolean function φ_f does not essentially depend on the last n variables if and only if f preserves the binary relation $\sigma = \{(0,0), (0,b), (b,0), (b,b), (a,a), (a,1), (1,1), (1,a)\} \subseteq D^2$. Thus, a quasi-De Morgan function is a Zhegalkin polynomial on the set D if and only if it preserves the binary relation σ.

From Theorem 4 we deduce that the term functions of the four-element Boolean algebra $(D; \{+, \cdot, {}', 0, 1\})$ and only they can be represented as Zhegalkin polynomials on the set D. This result also follows from the following equalities: $u' = 1 \oplus u$, $u \in D$ and $u + v = (u' \cdot v')' = 1 \oplus (1 \oplus u) \cdot (1 \oplus v) = u \oplus v \oplus u \cdot v$, $u, v \in D$. And these equalities give us an algorithm which, given a quasi-De Morgan function, gives its representation as a Zhegalkin polynomial on the set D.

A quasi-De Morgan function f is called *monotone* if it preserves the order of the lattice $(D; \{+, \cdot\})$.

It is easy to see that if f and g are monotone De Morgan functions then $f \cdot g$ and $f + g$ are monotone, too. Hence, all De Morgan functions of the following form are monotone De Morgan functions:

$$\sum_{s\in S} \prod_{i\in s} x_i, \qquad (5)$$

where $S \subseteq 2^{\{1,\ldots,n\}}$ is an antichain. In the proof of the following theorem we also show that all monotone De Morgan functions can be presented in the form (5).

Theorem 8. *A De Morgan function $f(x_1,\ldots,x_n)$ is monotone if and only if the corresponding Boolean function $\varphi_f(t_1,\ldots,t_{2n})$ does not essentially depend on the variables $t_{n+1},\ldots,t_{2n}$.*

Now we deduce the following result.

Corollary 3. *A De Morgan function can be represented as a Zhegalkin polynomial on the set D if and only if it is monotone.*

The idea of De Morgan and quasi-De Morgan functions is also applicable in quantum computation, quantum information theory, quantum logic and the theory of quantum computers.

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