On free A-bisemigroups

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ABSTRACT

In this paper, the concept of the A-bisemigroup is introduced and the construction of the free A-bisemigroup is described. (The A-bisemigroup is the duplex satisfying the additional identities.)

Keywords

Bisemigroup, free algebra, canonical form.

1. INTRODUCTION

The concepts of dimonoid and dialgebra were introduced in [1]. Dimonoid is a set with two binary associative operations satisfying the additional identities. Dialgebra is a linear analogue of the dimonoid. One of the first results about dimonoids is the description of the free dimonoid generated by the given set. Using properties of the free dimonoid, the free dialgebras were described and the cohomologies of dialgebras were studied in [1]. In [2], using the concept of dimonoids, the concept of unileteral dirings was introduced and the basic properties of dirings were studied. In [3, 4] free dimonoids and free commutative dimonoids were described. In [5], the concept of the duplex (which generalizes the concept of a dimonoid) and the construct of free duplexes were introduced. In [6, 7], the concept of Boolean bisemigroups (which generalizes the concept of the Boolean algebra) was introduced and a Stone-type theorem was proved (cf. [8, 9, 10]).

The concept of the 0-dialgebra was introduced in [11]. The 0-dialgebra under the field, F, is a vector space under F with two binary operations, $\dashv$ and $\vdash$, such that

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z, z \dashv (x \vdash y) = z \dashv (x \dashv y).$$

In this paper, the concept of the A-bisemigroup is introduced and the construction of the free A-bisemigroup is described. (The A-bisemigroup is the duplex satisfying the additional identities.)

2. AUXILIARY RESULTS

Definition 1. *The algebra, $(\mathcal{A}, \dashv, \vdash)$, is called A-bisemigroup if it satisfies the following identities:*

$$\begin{aligned}
&(A1)\ (x \dashv y) \dashv z = x \dashv (y \dashv z),\\
&(A2)\ (x \dashv y) \dashv z = x \dashv (y \vdash z),\\
&(A3)\ (x \dashv y) \vdash z = x \vdash (y \vdash z),\\
&(A4)\ (x \vdash y) \vdash z = x \vdash (y \vdash z).
\end{aligned}$$

Definition 2. *The map, $f : \mathcal{A}_1 \to \mathcal{A}_2$, between the A-bisemigroups, $\mathcal{A}_1$ and $\mathcal{A}_2$, is called homomorphism if $f(x \dashv y) = f(x) \dashv f(y)$, $f(x \vdash y) = f(x) \vdash f(y)$ for all $x, y \in \mathcal{A}_1$. The bijective homomorphism between the A-bisemigroups is called isomorphism.*

Lemma 1. *Let $(\mathcal{A}, \dashv, \vdash)$ be an A-bisemigroup. Then for any term, $t = t(x_1, \ldots, x_n)$, and for $x \in \mathcal{A}$ holds:*

(i) $x \dashv t = x \dashv x_1 \dashv \ldots \dashv x_n$,

(ii) $t \vdash x = x_1 \vdash \ldots \vdash x_n \vdash x$.

Proof. Prove (i) by induction on n. If $n = 1, 2$ then the statement is obvious. Let it be true for $n < k$. Any term, $t = t(x_1, \ldots, x_n)$, can be written as $t = t_1 * t_2$, where $* \in \{\dashv, \vdash\}$, $t_1 = t_1(x_1, \ldots, x_{k_1}), t_2 = t_2(x_{k_1+1}, \ldots, x_k)$, $0 < k_1 < k$. If $* = \dashv$, then

$$\begin{gathered}
x \dashv t = x \dashv (t_1 \dashv t_2) \overset{(A1)}{=} (x \dashv t_1) \dashv t_2 = \\
= (x \dashv x_1 \dashv \ldots \dashv x_{k_1}) \dashv t_2 = \\
= (x \dashv x_1 \dashv \ldots \dashv x_{k_1}) \dashv x_{k_1+1} \dashv \ldots \dashv x_k = \\
= x \dashv x_1 \dashv \ldots \dashv x_k.
\end{gathered}$$

If $* = \vdash$, then

$$\begin{gathered}
x \dashv t = x \dashv (t_1 \vdash t_2) \overset{(A2)}{=} (x \dashv t_1) \dashv t_2 = \\
(x \dashv x_1 \dashv \ldots \dashv x_{k_1}) \dashv t_2 = \\
= (x \dashv x_1 \dashv \ldots \dashv x_{k_1}) \dashv x_{k_1+1} \dashv \ldots \dashv x_k = \\
= x \dashv x_1 \dashv \ldots \dashv x_k.
\end{gathered}$$

Hence, (i) holds for $n = k$. (ii) is proved analogically. $\square$

Let e be an arbitrary symbol; introduce the following sets:

$$\begin{gathered}
I^1 = \{e\}, \quad I^n = \{0,1\}^{n-1} = \\
= \{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1}) : \varepsilon_k \in \{0,1\}, k = \overline{1, n-1}\},\ n > 1,
\end{gathered}$$

$$I = \bigcup_{n \geq 1} I^n.$$

Definition 3. *Let $(\mathcal{A}, \dashv, \vdash)$ be an A-bisemigroup. For any $x_1, x_2, \ldots, x_n \in \mathcal{A}$ and for $\varepsilon \in I^n$ define the element:*

$$x_1 x_2 \ldots x_n \varepsilon \in \mathcal{A} \qquad (\star),$$

by induction on $n \geq 2$ in the following way:

1. $x_1 e = x_1$,

2. $x_1 x_2 \dots x_n(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2}, 0) =$
$x_1 \vdash x_2 \dots x_n(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2})$,
$x_1 \dots x_{n-1} x_n(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2}, 1) =$
$= x_1 \dots x_{n-1}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2}) \dashv x_n$.

Particularly, if $\varepsilon = (\overbrace{1, 1, \dots, 1}^{n-1})$ then $x_1 \dots x_n \varepsilon = x_1 \dashv \dots \dashv x_n$; if $\varepsilon = (\overbrace{0, 0, \dots, 0}^{n-1})$, then $x_1 \dots x_n \varepsilon = x_1 \vdash \dots \vdash x_n$.

Lemma 2. In any A-bisemigroup the following identities hold:

$$x_1 x_2 \dots x_n(\varepsilon_1, \dots, \varepsilon_{n-1}) \dashv y_1 y_2 \dots y_m(\theta_1, \dots, \theta_{m-1}) =$$
$$= x_1 x_2 \dots x_n y_1 y_2 \dots y_m(\varepsilon_1, \dots, \varepsilon_{n-1}, \overbrace{1, 1, \dots, 1}^{m}),$$

$$x_1 x_2 \dots x_n(\varepsilon_1, \dots, \varepsilon_{n-1}) \vdash y_1 y_2 \dots y_m(\theta_1, \dots, \theta_{m-1}) =$$
$$= x_1 x_2 \dots x_n y_1 y_2 \dots y_m(\theta_1, \dots, \theta_{m-1}, \overbrace{0, 0, \dots, 0}^{n}).$$

Proof.

$$x_1 x_2 \dots x_n(\varepsilon_1, \dots, \varepsilon_{n-1}) \dashv y_1 y_2 \dots y_m(\theta_1, \dots, \theta_{m-1})$$
$$\overset{(i)}{=} x_1 x_2 \dots x_n(\varepsilon_1, \dots, \varepsilon_{n-1}) \dashv y_1 \dashv y_2 \dashv \dots \dashv y_m$$
$$\overset{2.}{=} x_1 x_2 \dots x_n y_1 y_2 \dots y_m(\varepsilon_1, \dots, \varepsilon_{n-1}, \overbrace{1, 1, \dots, 1}^{m}),$$

$$x_1 x_2 \dots x_n(\varepsilon_1, \dots, \varepsilon_{n-1}) \vdash y_1 y_2 \dots y_m(\theta_1, \dots, \theta_{m-1})$$
$$\overset{(ii)}{=} x_1 \vdash x_2 \vdash \dots \vdash x_n \vdash y_1 y_2 \dots y_m(\theta_1, \dots, \theta_{m-1})$$
$$\overset{2.}{=} x_1 x_2 \dots x_n y_1 y_2 \dots y_m(\theta_1, \dots, \theta_{m-1}, \overbrace{0, 0, \dots, 0}^{n}).$$

□

All terms of the A-bisemigroup can be described by the elements of the form, $(\star)$; namely, we can show that any term can be reduced to $(\star)$.

Theorem 1. Let $t = t(x_1, \dots, x_n)$ be a term in the A-bisemigroup. Then there is such $\varepsilon \in I^n$ that

$$t = x_1 x_2 \dots x_n \varepsilon.$$

By virtue of theorem 1, any term in the A-bisemigroup can be reduced to the form, $(\star)$, which we call canonical form. For example, for the term, $((x_1 \dashv x_2) \vdash (x_3 \dashv x_4)) \dashv (x_5 \vdash x_6)$, the canonical form is:

$$((x_1 \dashv x_2) \vdash (x_3 \dashv x_4)) \dashv (x_5 \vdash x_6) =$$
$$= (x_1 x_2(1) \vdash x_3 x_4(1)) \dashv x_5 x_6(0) =$$
$$= x_1 x_2 x_3 x_4(1, 0, 0) \dashv x_5 x_6(0) =$$
$$= x_1 x_2 x_3 x_4 x_5 x_6(1, 0, 0, 1, 1).$$

Define the operations, $\dashv$ and $\vdash$, on I in the following way:

$$(\varepsilon_1, \dots, \varepsilon_{n-1}) \dashv (\theta_1, \dots, \theta_{m-1}) = (\varepsilon_1, \dots, \varepsilon_{n-1}, \overbrace{1, 1, \dots, 1}^{m}),$$

$$(\varepsilon_1, \dots, \varepsilon_{n-1}) \vdash (\theta_1, \dots, \theta_{m-1}) = (\theta_1, \dots, \theta_{m-1}, \overbrace{0, 0, \dots, 0}^{n}).$$

To show that the canonical form for any term is unique we need the following lemma:

Lemma 3. The algebra, $(I, \dashv, \vdash)$, is an A-bisemigroup.

Proof. The axioms, $(A1), (A2), (A3), (A4)$, are checked directly. □

Lemma 4. In the algebra, $(I, \dashv, \vdash)$, for any $\varepsilon \in I^n$, we have:

$$\underbrace{ee \dots e}_{n}\varepsilon = \varepsilon.$$

Proof. Prove by induction on n. If $n = 1$, then the statement follows from definition 2.3. Let it be true for $n = k$ and let $\varepsilon \in I^{k+1}$.

If $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{k-1}, 0) = e \vdash \varepsilon'$, where $\varepsilon' = (\varepsilon_1, \dots, \varepsilon_{k-1})$, then

$$\underbrace{ee \dots e}_{k+1}\varepsilon = e \vdash \underbrace{ee \dots e}_{k}\varepsilon' = e \vdash \varepsilon' = \varepsilon.$$

If $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{k-1}, 1) = \varepsilon' \dashv e$, where $\varepsilon' = (\varepsilon_1, \dots, \varepsilon_{k-1})$, then

$$\underbrace{ee \dots e}_{k+1}\varepsilon = \underbrace{ee \dots e}_{k}\varepsilon' \dashv e = \varepsilon' \dashv e = \varepsilon.$$

□

From definitions of the operations, $\dashv, \vdash$, it follows:

Lemma 5. If $\alpha \in I^n, \theta \in I^m$, then $\alpha \dashv \theta, \alpha \dashv \theta \in I^{n+m}$.

Now we can formulate the uniqueness of the canonical form.

Theorem 2. For any term in the A-bisemigroup the canonical form is unique.

3. MAIN RESULT

Let's turn to the construction of the free A-bisemigroup. Let X be an arbitrary and nonempty set, $n \in \mathcal{N}$. Denote:

$$Y_n = X^n \times I^n, \qquad n \in \mathcal{N},$$

where $X^n = \underbrace{X \times X \times \dots \times X}_{n} = \{(x_1, x_2, \dots, x_n) : x_k \in X, k = \overline{1, n}\}$,

$$\mathcal{A}(X) = \bigcup_{n \geq 1} Y_n.$$

For convenience, the elements of $\mathcal{A}(X)$ are denoted by $(x_1, x_2, \dots, x_n)\varepsilon$, where $\varepsilon \in I^n$, instead of $((x_1, x_2, \dots, x_n), \varepsilon)$, and we consider the sets, $X \times I^1$ and X, being the same, that is,we identify the symbol, $x \in X$, with the element, $xe \in \mathcal{A}(X)$. Define the operations, $\dashv, \vdash$ on $\mathcal{A}(X)$ in the following way:

$$(x_1, x_2, \dots, x_k)\varepsilon \dashv (x_{k+1}, x_{k+2}, \dots, x_l)\theta =$$
$$= (x_1, x_2, \dots, x_l)\varepsilon \dashv \theta,$$

$$(x_1, x_2, \dots, x_k)\varepsilon \vdash (x_{k+1}, x_{k+2}, \dots, x_l)\theta =$$
$$= (x_1, x_2, \dots, x_l)\varepsilon \vdash \theta.$$

Theorem 3. The binary algebra, $(\mathcal{A}(X), \dashv, \vdash)$, is a free A-bisemigroup with the system of free generators, X.

Give another description of the free A-bisemigroup. Let $F[X]$ be a free semigroup with the system of the free generators, X. For any word, $\omega \in F[X]$, we denote the length of ω by $|\omega|$. Define the operations, $\dashv, \vdash$, on the set:

$$FA = \{(\omega, \varepsilon) : \omega \in F[X], \varepsilon \in I^{|\omega|}\}$$

in the following way:

$$(\omega_1, \varepsilon) \dashv (\omega_2, \theta) = (\omega_1\omega_2, \varepsilon \dashv \theta),$$

$$(\omega_1, \varepsilon) \vdash (\omega_2, \theta) = (\omega_1\omega_2, \varepsilon \vdash \theta),$$

$(\omega_1, \varepsilon), (\omega_2, \theta) \in FA$. It is easy to verify that the binary algebra, $(FA, \dashv, \vdash)$, is an A-bisemigroup, which we denote by $FA[X]$.

Theorem 4. The A-bisemigroups $(\mathcal{A}(X), \dashv, \vdash)$ and $FA[X]$ are isomorphic.

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