On Some Semantics of the Symmetric Constructive Logic

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ABSTRACT

A semantics for the Symmetric Constructive Logic [1] is introduced. This semantics is based on methods similar to those developed for other logical systems in [3] – [5]. The following theorems are formulated.

(1) Any predicate formula deducible in symmetric constructive predicate calculus is identically true concerning the introduced semantics.

(2) Some predicate formulas not deducible in the mentioned calculus are not identically true concerning the introduced semantics.

Relations of this semantics and the semantics of logical system described in [4] are considered.

Keywords

Symmetric constructive logic, recursively enumerable set, RES-ideal, RED-ideal, REFS-ideal, SREFS-ideal, predicate formula, predicate calculus.

The logical system of the Symmetric Constructive Logic (shortly, SCL) is defined in [1]. A semantics of SCL based on the notion of the symmetric realizability of predicate formulas is considered in [1]; some modification of this semantics is described in [2]. Below a semantics for SCL based on the properties of recursively enumerable sets (shortly, RES) is introduced. In the definition of this semantics some methods are used similar to those considered in [3] - [5].

We suppose that the reader is familiar with the theory of recursive functions ([7] - [10]), with the classical and constructive (intuitionistic) predicate logic ([10] – [12]), and with the properties of constructive mathematics ([13] – [15]). By N we denote the set of all non-negative integers, N={0,1,2,...}.

The notion of n-dimensional RES (where $n \geq 1$), as well as the operations of union (∪) and intersection (∩) of RESes are defined in a usual way ([6] - [10]). The Cartesian product $\alpha \times \beta$ of RESes α and β, the k^{th} Cartesian power α^k of a RES α, the projection along the i^{th} coordinate $\downarrow_i^n(\alpha)$ for an n-dimensional RESα (where $1 \leq i \leq n$), the generalization along the i^{th} coordinate $\uparrow_i^n(\alpha)$ for an n-dimensional RES α (where $1 \leq i \leq n$) are defined as in ([3] – [6]). The operation of transposition $\mathrm{T}_{ij}^n(\alpha)$ of the i^{th} and j^{th} coordinates in an n-dimensional RES α (where $1 \leq i,j \leq n$) as well as the operation of substitution $Sub_{ij}^n(\alpha)$ of the i^{th} co-ordinate for the j^{th} in an n-dimensional RES α (where $1 \leq i,j \leq n$) are defined as in ([3] – [6]). By V^n (correspondingly, Λ^n) we denote the set of all n-tuples (x_1, x_2, ..., x_n), where $x_i \in N$ for $1 \leq i \leq n$ (correspondingly, the empty set which is considered as a subset of V^n), we write V and Λ instead of V^1 and Λ^1.

The notion of n-dimensional RES-ideal is defined as in ([4] – [5]). Let us recall some notions connected with it. n-dimensional RES-ideal is defined as a constructive non-empty set Δ of n-dimensional RESes satisfying the following conditions:

(1) if $\alpha \in \Delta$ and $\beta \subseteq \alpha$, then $\beta \in \Delta$;

(2) if $\alpha \in \Delta$ and $\beta \in \Delta$, then $\alpha \cup \beta \in \Delta$.

The notion of constructive set is interpreted as in [13] – [15], namely, only such sets are considered which can be defined by formulas in some fixed meta-language; we consider the constructive (intuitivnistic) arithmetic in its informal interpretation as such meta-language.

The notions of a principal RES-ideal, of a complete RES-ideal, of a null Res-ideal, of a RES-ideal generated by a given non-empty set of n-dimensional RESes are defined as in [5]. Two n-dimensional RES-ideals Δ_1 and Δ_2 are said to be disjoint (cf. [5]) if $\alpha \cap \beta = \Lambda^n$ for any $\alpha \in \Delta_1$ and $\beta \in \Delta_2$.

n-dimensional RED-ideal is defined as any pair (Δ_1, Δ_2) of disjoint n-dimensional RES-ideals. If Δ is an n-dimensional RED-ideal (Δ_1, Δ_2) then Δ_1 and Δ_2 will be called correspondingly the *positive* and *negative* components of Δ; they will be denoted by Δ^+ and Δ^-. A RED-ideal (Δ^+, Δ^-) is said to be a *principal* one if Δ^+ and Δ^- are *principal* RES-ideals. A RED-ideal (Δ^+, Δ^-) is said to be a *complete* one if Δ^+ is a *complete* RES-ideal and Δ^- is a *null* RES-ideal. It is said to be *null* RED-ideal if Δ^+ is a *null* RES-ideal and Δ^- is a *complete* RES-ideal.

Predicate formula based on the operations &,∨, ⊃, ¬, ∀, ∃ is defined in an usual way ([8] – [11]); we consider predicate formulas which do not contain functional symbols and symbols of constants. The symbols T (truth), F (falsity), U (uncertainty) are considered as elementary formulas (cf. [4]). All the notions connected with the predicate formulas (in particular, the notion of *index majorant* for a given predicate formula) are defined as in ([3]-[5]).

Below we shall define the semantics of predicate formulas in the framework of the symmetric constructive logic (shortly, the *SCL-semantics*). We shall give the definition of this semantics using the theory of fuzzy recursively enumerable sets. Let us note that the definition of SCL-semantics can be given independently of the mentioned theory, however the using of this theory gives the possibility to formulate the necessary definitions in a comparatively short and simple way.

Let us recall some definitions (cf. [3] – [6]). N-dimensional recursively enumerable fuzzy set (shortly, REFS), where $n \geq 1$, is defined as a recursively enumerable set of (n + 1)-tuples (x_1, x_2, ... ,x_n, ε) such that $x_i \in N$ for $1 \leq i \leq n$, and ε is a binary rational number having the form $k / 2^m$, where $k \in N$, $m \in N$ and $0 \leq k / 2^m \leq 1$. If α is an n-dimensional RES, then the *fuzzy image* of α is defined as an n-dimensional REFS β satisfying the following conditions (cf. [5]):

(1) if (x_1, x_2, ..., x_n) $\in \alpha$ then (x_1, x_2, ... ,x_n, ε) $\in \beta$ for any binary rational ε such that $0 \leq \varepsilon < 1$;

(2) if (x_1, x_2, ..., x_n) $\notin \alpha$ then (x_1, x_2, ... ,x_n, ε) $\in \beta$ only for $\varepsilon = 0$.

The fuzzy image of a RES α we shall denote by *FIM(α)*. If β is an n-dimensional REFS, then the *support* of β is defined as an *n*-dimensional RES α such that $(x_1, x_2, ..., x_n) \in \alpha$ if and only if there exists $\varepsilon > 0$ such that $(x_1, x_2, ... ,x_n, \varepsilon) \in \beta$ (this definition differs only technically from the corresponding definition given in [5]). The support of REFS β we shall denote by *Supp*(β). It is easily seen that for any RES α the following statement holds: $Supp(FIM(\alpha)) = \alpha$.

Main notions connected with REFSes, in particular, the operations $\cup$, $\cap$, $\times$, $\downarrow_i^n$, $\uparrow_i^n$, T_{ij}^n, Sub^n_{ij}, the relation of *equivalence* between REFSes α_1 and α_2 (denoted by $\alpha_1 = \alpha_2$), the relation of *covering* of a REFS α_1 by a REFS α_2 (denoted by $\alpha_1 \subseteq \alpha_2$), the REFSes V^n and Λ^n are defined as in [3] and [5]. (Let us note that the notations V^n and Λ^n may be interpreted as notations of RESes and REFSes; the concrete sense of these notations will be seen from the context). The relation of *strong equivalence* between REFSes is defined as in [5]. Let us note that if α is an *n*-dimensional REFS, then the REFSes $FIM(Supp(\alpha))$ and α are equivalent but in general not strongly equivalent (see [5]). *n*-dimensional REFS-ideal and the main notions connected with it (in particular the notion of a *principal* REFS-ideal, of a *complete* REFS-ideal, of a *null* REFS-ideal, of a REFS-ideal, *generated* by a given set of REFSes) are defined as in [5]. Two *n*-dimensional REFS-ideals Δ_1 and Δ_2 are said to be *disjoint* if $\alpha_1 \cap \alpha_2 = \Lambda^n$ for any $\alpha_1 \in \Delta_1$ and $\alpha_2 \in \Delta_2$ (see [5]).

If Δ is an *n*-dimensional RES-ideal then its *fuzzy image* FIM(Δ) is defined as the set of all *n*-dimensional REFSes β satisfying the following condition: there exists $\Delta \in \alpha$ such that $\beta \subseteq$ FIM(α). If Δ is an *n*-dimensional REFS-ideal, then its *support* Supp(Δ) is defined as the set of all α having the form α = Supp(β) for some $\beta \in \Delta$. It is easily seen that FIM(Δ) is an *n*-dimensional REFS-ideal for any *n*-dimensional RES-ideal Δ. Similarly, Supp(Δ) is an *n*-dimensional RES-ideal for any *n*-dimensional REFS-ideal Δ. The notion of SREFS-ideal is defined as in [4]. Namely, *n*-dimensional SREFS-ideal is defined as any pair of disjoint *n*-dimensional REFS-ideals (Δ_1, Δ_2). The components Δ_1 and Δ_2 of a SREFS-ideal $\Delta = (\Delta_1, \Delta_2)$ are called correspondingly the *positive* and *negative* components of Δ; they are denoted by Δ^+ and Δ^-. Main notions connected with SREFS-ideals are defined as in [4]. If $\Delta = (\Delta^+, \Delta^-)$ is a SREFS-ideal then its *support* Supp(Δ) is defined as the pair (Supp(Δ^+), Supp(Δ^-)). If $\Delta = (\Delta^+, \Delta^-)$ is a RED-ideal then its *fuzzy image* FIM(Δ) is defined as the pair (FIM(Δ^+), FIM(Δ^-)). It is easily seen that Supp(Δ) is a RED-ideal for any SREFS-ideal Δ. Similarly, FIM (Δ) is a SREFS-ideal for any RED-idealΔ.

Now let us give the definition of SCL-semantics. Let *A* be a predicate formula which does not contain predicate symbols except the symbols p_1, p_2, ... , p_l having the dimensions correspondingly i_1, i_2, ... , i_l.

A *SCL-assignment* for *A* is defined as a correspondence assigning to any p_m, where $1 \leq m \leq l$, some i_m-dimensional RED-ideal. A SCL-assignment for *A* is said to be *principal* if all RED-ideals assigned to p_1, p_2, ... , p_l are principal RED-ideals.

Let φ be a SCL-assignment for *A*, *k* be an index majorant of *A*. We shall define the SCL-interpretation of *A* concerning a given SCL-assignment φ and a given index majorant *k* of *A* as a RED-ideal obtained by the following operations. We consider the correspondence ψ assigning to every predicate symbol p_m contained in *A* the SREFS-ideal FIM(Δ), where Δ is the RED-ideal assigned to p_m by the correspondence φ. It is easily seen that ψ is a SFCL*-assignment for *A* in the sense of corresponding definition given in [4]. The *SCL-interpretation* of *A* concerning a given SCL-assignment φ for *A* and a given index majorant *k* of *A* is defined as Supp($\Pi_{\psi,k}$), where $\Pi_{\psi,k}$ is the SFCL*-interpretation of *A* concerning ψ and *k* in the sense of the corresponding definition given in [4]. It is easily seen that the SCL-interpretation of *A* concerning any SCL-assignment φ for *A* and any sufficient great index majorant *k* of *A* is an *k*-dimensional RED-ideal. The formula *A* is said to be *strongly SCL-valid* (correspondingly, *weakly SCL-valid*) if for any SCL-assignment φ for *A* (correspondingly, for any principal SCL-assignment φ for *A*) and for any sufficiently great index majorant *k* of *A* the SCL-interpretation of *A* concerning φ and *k* is a complete RED-ideal. These notions define the semantics of predicate formulas in SCL.

We consider the calculus HSU' introduced in [4]; let us recall that this calculus is obtained from the symmetric constructive predicate calculus HSU defined in [16] by excluding predicate formulas containing functional symbols and symbols of constants.

Theorem 1.

Any predicate formula deducible in HSU' is strongly SCL-valid.

Theorem 2.

The formulas $(p(x_1) \supset (p(x_1) \supset q(x_1))) \supset (p(x_1) \supset q(x_1))$, $\neg(p(x_1) \ \& \ \neg p(x_1))$, $(p(x_1) \ \& \ \exists x_2(q(x_1, x_2))) \supset \exists x_2(p(x_1) \ \& \ q(x_1, x_2))$ are not weakly SCL-valid.

Note. The formulas considered in Theorem 2 are not deducible in HSU'. Theorem 1 may be deduced from Theorem 1 formulated in [4] using the relations between the logical system considered in [4], and SCL. However, the statement of Theorem 2 is a stronger one than the statement of Theorem 2 formulated in [4].

Using the relations between the systems mentioned above we can deduce Theorem 2 in [4] from Theorem 2 formulated here.

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