A Note on Long non-Hamiltonian Cycles in One Class of Digraphs

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ABSTRACT
Let $D$ be a strong digraph on $n \geq 4$ vertices. In [3, Discrete Applied Math., 95 (1999) 77–87]], J. Bang-Jensen, Y. Guo and A. Yeo proved the following theorem: if (*) $d(x) + d(y) \geq 2n - 1$ and $\min\{d^{+}(x) + d^{-}(y) , d^{+}(x) + d^{-}(y)\} \geq n - 1$ for every pair of non-adjacent vertices $x, y$ with a common in-neighbour or a common out-neighbour, then $D$ is hamiltonian. In this note we show that: if $D$ is not a directed cycle and satisfies the condition (*), then $D$ contains a cycle of length $n - 1$ or $n - 2$.

Keywords
Digraphs, cycles, Hamiltonian cycles, long non-Hamiltonian cycles.

1. INTRODUCTION AND TERMINOLOGY
We shall assume that the reader is familiar with the standard terminology on directed graphs (digraphs) and refer the reader to monograph of Bang-Jensen and Gutin [1] for terminology not discussed here. In this paper we consider finite digraphs without loops and multiple arcs. For a digraph $D$, we denote by $V(D)$ the vertex set of $D$ and by $A(D)$ the set of arcs in $D$. Often we will write $D$ instead of $A(D)$ and $V(D)$. The arc of a digraph $D$ directed from $x$ to $y$ is denoted by $xy$. For disjoint subsets $A$ and $B$ of $V(D)$ we define $A \rightarrow B$ as the set $\{xy \in A(D)x \in A, y \in B\}$ and $A \leftarrow B = A \rightarrow B \cup B \rightarrow A$. If $x \in V(D)$ and $A = \{x\}$ we write $x$ instead of $\{x\}$. The out-neighbourhood of a vertex $x$ is the set $N^{+(x)} = \{y \in V(D)/xy \in A(D)\}$ and $N^{-(x)} = \{y \in V(D)/yx \in A(D)\}$ is the in-neighbourhood of $x$. Similarly, if $A \subseteq V(D)$ then $N^{+(x,A)} = \{y \in A/xy \in A(D)\}$ and $N^{-(x,A)} = \{y \in A/yx \in A(D)\}$. We call the vertices in $N^{+(x)}$, $N^{-(x)}$, the out-neighbours and in-neighbours of $x$. The out-degree of $x$ is $d^{+(x)} = |N^{+(x)}|$ and $d^{-(x)} = |N^{-(x)}|$ is the in-degree of $x$. The out-degree and in-degree of $x$ we call its semi-degrees. Similarly, $d^{+(x,A)} = |N^{+(x,A)}|$ and $d^{-(x,A)} = |N^{-(x,A)}|$. The degree of the vertex $x$ in $D$ is defined as $d(x) = d^{+(x)} + d^{-(x)}$ (similarly, $d(x,A) = d^{+(x,A)} + d^{-(x,A)}$). The subdigraph of $D$ induced by a subset $A$ of $V(D)$ is denoted by $A$. The path (respectively, the cycle) consisting of the distinct vertices $x_{1}, x_{2}, \ldots , x_{m}$ ($m \geq 2$) and the arcs $x_{i}x_{i+1}, i \in [1, m-1]$ (respectively, $x_{i}x_{i+1}, i \in [1, m-1]$, and $x_{m}x_{1}$), is denoted $x_{1}x_{2}\ldots x_{m}$ (respectively, $x_{1}x_{2}\ldots x_{m}x_{1}$). For a cycle $C_{k} = x_{1}x_{2}\ldots x_{m}$, the subscripts considered modulo $k$, i.e. $x_{i} = x_{i}$ for every $s$ and $i$ such that $i \equiv s \pmod{k}$. If $P$ is a path containing a subpath from $x$ to $y$ we let $P[x,y]$ denote that subpath. Similarly, if $C$ is a cycle containing vertices $x$ and $y$, $C[x,y]$ denotes the subpath of $C$ from $x$ to $y$. A digraph $D$ is strongly connected (or just strong) if there exists a path from $x$ to $y$ and a path from $y$ to $x$ in $D$ for every choice of distinct vertices $x, y$ of $D$. We will denote the complete bipartite digraph with partite sets of cardinalities $p, q$ by $K_{p,q}$. Two distinct vertices $x$ and $y$ are adjacent if $xy \in A(D)$ or $yx \in A(D)$ (or both). We denote by $a(x,y)$ the number of arcs between the vertices $x$ and $y$. In particular, $a(x,y) = 0$ (respectively, $a(x,y) \neq 0$) means that $x$ and $y$ are not adjacent (respectively, are adjacent).

For integers $a$ and $b$, $a \leq b$, let $[a, b]$ denote the set of all integers which are not less than $a$ and are not greater than $b$. The digraph $D$ is hamiltonian (is pancyclic, respectively) if it contains a hamiltonian cycle, i.e. a cycle of length $|V(D)|$ (contains a cycle of length $m$ for any $3 \leq m \leq |V(D)|$).

Meyniel [12] proved the following theorem: if $D$ is a strong digraph on $n \geq 2$ vertices and $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices in $D$, then $D$ is hamiltonian (for short proofs of Meyniel’s theorem see [4, 13]).

Thomassen [15] (for $n = 2k + 1$) and Darbinyan [6] (for $n = 2k$) proved: if $D$ is a digraph on $n \geq 5$ vertices with minimum degree at least $n - 1$ and with minimum semi-degree at least $n/2 - 1$, then $D$ is hamiltonian (unless some extremal cases).

In each above mentioned theorems (as well as, in theorems Ghoula-Houri [10], Woodall [16], Manoussakis [11]) imposes a degree condition on all pairs of non-adjacent vertices (on all vertices). Bang-Jensen, Gutin, Li, Guo and Yeo [2, 3] obtained sufficient conditions for hamiltonicity of digraphs in which degree conditions requiring only for some pairs of non-adjacent vertices. Namely, they proved the following theorems (in all three theorems $D$ is a strong digraph on $n \geq 2$ vertices).

**Theorem A** [1, 2]. If $\min\{d(x), d(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for every pair of non-adjacent vertices $x, y$ with a common in-neighbour, then $D$ is hamiltonian.

**Theorem B** [1, 2]. If $\min\{d^{+(x)} + d^{-}(y), d^{+(x)} + d^{-(y)}\} \geq n$ for every pair of non-adjacent vertices $x, y$ with a common out-neighbour or a common in-neighbour, then $D$ is hamiltonian.

**Theorem C** [3]. If $\min\{d^{+(x)} + d^{-}(y), d^{+(x)} + d^{-(y)}\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for every pair of non-
adjacent vertices \(x, y\) with a common out-neighbour or a common in-neighbour, then \(D\) is hamiltonian. Note that Theorem C generalizes Theorem B. In [9, 14, 5, 7] it was shown that if the strong digraph \(D\) satisfies the condition of the theorem of Ghouila-Houri [10] (Woodall [16], Meyniel [12], Thomassen and Darbinyan [15, 6]), then \(D\) is pancyclic (unless some exceptional cases, which are characterized). In [8], we posed the following problem:

**Problem.** Characterize those digraphs which satisfy the conditions of Theorem A (B, C), but are not pan-cyclic.

In [8], we have shown that:

(i) if a strong digraph \(D\) satisfies the condition of Theorem A and the minimum semi-degree of \(D\) at least two; or

(ii) if a strong digraph \(D\) is not a directed cycle and satisfies the condition of Theorem B, then either \(D\) contains a cycle of length \(n - 1\) or \(n\) is even and \(D\) is isomorphic to complete bipartite digraph or to complete bipartite digraph minus one arc.

In [8], we also posed the following

**Conjecture.** Let a digraph \(D\) on \(n \geq 4\) vertices satisfy the conditions of Theorem C. Then \(D\) contains a cycle of length \(n - 1\) maybe except some digraphs which have a ”simple” structure.

Support for the our conjecture, in this note by using the proof of Theorem C (Theorem 3.1, [3]), we show that: if \(D\) is not a directed cycle and satisfies the conditions of Theorem C, then \(D\) contains a cycle of length \(n - 1\) or \(n - 2\).

2. PRELIMINARIES

The following well-known simple lemmas are the basis of our results and other theorems on directed cycles and paths in digraphs. They will be extensively used in the proofs of our results.

**Lemma 1** [9]. Let \(D\) be a digraph on \(n \geq 3\) vertices containing a cycle \(C_m, m \in \{2, n - 1\}\). Let \(x\) be a vertex not contained in this cycle. If \(d(x, C_m) \geq m + 1\), then \(D\) contains a cycle \(C_k\) for all \(k \in \{2, m + 1\}\).

**Lemma 2** [4]. Let \(D\) be a digraph on \(n \geq 3\) vertices containing a path \(P := \{x_1 x_2 \ldots x_m \mid m \in \{2, n - 1\}\}\) and let \(x\) be a vertex not contained in this path. If one of the following conditions holds:

(i) \(d(x, P) \geq m + 2\);

(ii) \(d(x, P) \geq m + 1\) and \(x \in P \) or \(x, x_1 \in D\);

(iii) \(d(x, P) \geq m\), \(x \in P\), and \(x, x_1 \in D\) and \(x_1 \in P\), then there is an \(i \in \{1, m - 1\}\) such that \(x, x_i, x_{i+1} \in D\), where \(m\) is a vertex selected into \(P\) or the arc \(x, x_{i+1}\) is a partner of \(x\) on \(P\).

**Lemma 3** [2]. Let \(P := x_1 x_2 \ldots x_m\) be a path in \(D\) and let \(x, y\) be vertices of \(V(D) - V(P)\) (possibly \(x = y\)). If there do not exist consecutive vertices \(x_i, x_{i+1}\) on \(P\) such that \(x, y, x_{i+1}\) are arcs of \(D\), then \(d^-(x, P) + d^+(y, P) \leq m + 1\), where \(\epsilon = 1\) if \(x_1 \in D\) and \(0\), otherwise.

3. MAIN RESULT

Let \(C\) be a cycle in digraph \(D\). For the cycle \(C\), a \(C\)-bypass is an \((x, y)\)-path \(P\) of length at least two with both end-vertices \(x\) and \(y\) on \(C\) and no other vertices on \(C\). The length of the path \(C[x, y]\) is the gap of \(P\) with respect to \(C\).

If \(\{x, y\}\) is a pair of non-adjacent vertices with a common in-neighbour or a common out-neighbour, then in the proof of the theorem we say that \(\{x, y\}\) is a good pair.

In the proof of our theorem we use (mainly) the notations which are used in the proof of Theorem C (Theorem 3.1, [3]).

**Theorem.** Let \(D\) be a strong digraph with \(n \geq 2\) vertices, which is not a directed cycle. Suppose that \(\min(d^+(x) + d^-(y), d^+(x) + d^-(y)) \geq n - 1\) and \(d(x) + d(y) \geq 2n - 1\) for every pair of non-adjacent vertices \(x, y\) with a common out-neighbour or a common in-neighbour, then \(D\) contains a cycle of length \(n - 2\) or \(n - 1\).

**Proof.** Suppose, to the contrary, that \(D\) contains no cycles of length \(n - 2\) or \(n - 1\). Let \(C := x_1 x_2 \ldots x_n x_1\) be a longest non-hamiltonian cycle in \(D\). Then \(3 \leq m \leq n - 3\) and let \(R := V(D) - V(C)\). Observe that if \(y \in V(C)\), then \(y\) has no partner on \(C\). We shall use this often without an explicit reference. For the digraph \(D\) provided that \(D\) is not hamiltonian, in [3] (Theorem 3.1), J. Bang- Jensen, Y. Guo and A. Yeo proved the following Claims 1 and 2.

**Claim 1.** Let \(y\) be a vertex of \(R\). If \(x_o \neq x_3, x_o y, x_3 y \in D\) and \(\ell(y, C \setminus V(C[x, y], x_1, x_0)) = 0\), then the following holds:

\[|V(C')| \geq 1, \quad d(y, C) = d(y, C') = |C'| + 1, \quad \text{(1)}\]

\[d^+(x_{j-1}, C') + d^+(x_{j+1}, C') = |C'| + 1, \quad \text{(2)}\]

\[d(y, R) + d^+(x_{j-1}, R) + d^-(x_{j+1}, R) = 2(n - m - 1), \quad \text{(3)}\]

\[d^+(x_{j-1}, C') = d(x_{j+1}, C') = |C'| - 1, \quad \text{(4)}\]

where \(C' := C[x_{j+1}, x_{j-1}]\) and \(C' := C[x_{j-1}, x_0]\).

**Claim 2.** \(D\) contains a \(C\)-bypass. \(\square\) Note that Claims 1 and 2 also are true if in \(D\) a longest non-hamiltonian cycle has length at most \(n - 3\) (the proofs are just the same).

From (4) it follows that if \(|C'| \geq 2\), then \(P := x_{j-1} x_{j+2} \ldots x_{j-2} x_{j+1}\) is a hamiltonian \((x_{j-1}, x_{j+1})\)-path in \((C')\). Therefore, similarly (2), we obtain (Lemma 3)

\[d^-(x_{j-1}, C') + d^+(x_{j+1}, C') = |C'| + 1. \quad \text{(5)}\]

Combining this last inequality with (2) yields

\[d(x_{j-1}, C') + d(x_{j+1}, C') \leq 2|C'| + 2. \quad \text{(5)}\]

We now prove the following claim:

**Claim 3.** Let \(x_m y, x_{m+1} \in D\) be a \(C\)-bypass and \(A(y, C[x, x_1]) = 0\). Then \(\gamma \geq 3\).

**Proof.** Suppose that \(\gamma \leq 2\). Let now \(C'' := C[x_{m+1}, x_m]\). We shall consider the cases \(\gamma = 1, \gamma = 2\) separately.

**Case 1.** \(\gamma = 1\). Then similarly (1) and (3) we have \(d(y, C') \leq m\) and \(d(x_1, C') \leq m\) and \(d(y, R) + d(x_1, R) \leq 2(n - m - 1)\). Therefore, since \(\{x, x_1\}\) is a good pair and \(|C''| = m - 1\), we have

\[2n - 1 \leq d(x) + d(x_1) = d(y, R \cup C') + d(x_1, R \cup C') \leq 2(n - m - 1) + 2|C''| + 2 = 2n - 2, \quad \text{a contradiction.}\]

**Case 2.** \(\gamma = 2\). Then, since \(|R| \geq 3\), for any \(i \in \{1, 2\}\) we obtain that \(d(y, R) + d(x_1, R) \leq 2(n - m - 1)\). Since \(\{x, x_1\}\) is a good pair and (1), it follows that

\[2n - 1 \leq d(y) + d(x_1) \leq 2(n - m - 1) + d(y, C') + d(x_1, C') \leq 2(n - m - 1) + |C''| + 3 + d(x_1, C'). \quad \text{(6)}\]

From this we obtain that \(d(x_1, C''') \geq m = |C''| + 2\). Hence, by Lemma 2, the vertex \(x_1 (x_2)\) has a partner
on $C'$. Therefore there is a $(x_1, x_m)$-path with vertex set $V(C)$. This path with the vertex $y$ forms a non-
hamiltonian cycle longer than $C$. Claim 3 is proved. □

Let $P := u_1 u_2 \ldots u_s$ $(s \geq 3)$ be a $C$-bypass with minimum gap among the gaps of all $C$-bypasses. Assume
w.l.o.g. that $P$ is minimal with respect to the minimum gap
and let $u_i := x_1, u_j := x_w$ with $2 \leq \gamma \leq m$.

In the following we suppose, further, that $\gamma = 2$ (the
proof for the case $\gamma = 3$ is same as the proof of Theo-
rem C (Theorem 3.1 [3])).

Then $R = \{u_2, u_3, \ldots, u_{s-1}\}$, $s \geq 5$ and for any pair of
$i, j$ with $2 \leq i < j \leq s - 1$

\[ u_{i,j} \in D \quad \text{if and only if} \quad j = i + 1. \]  
(6)

Since $|R| \geq 3$ and $C$ is a longest non-hamiltonian cycle
in $D$, it is easy to see that

\[ x_1 u_{s-1} \notin D, \ u_2 x_2 \notin D \]

and

\[ d^-(u_2, \{x_{m-1}, x_3\}) = d^+(u_{s-1}, \{x_3, x_4\}) = 0. \]  
(7)

**Case 1.** $x_2 u_2 \notin D$ and there is an $i \in [3, m]$ such
that $x_2 u_i \notin D$. Then by (7) we have $a(u_2, u_2) = 0$, $\leq i \leq m - 2$ and by Claim 3, $d^+(u_2, \{x_3, x_4\}) = 0$.

Assume that $d^+(u_2, C[x_3 \cup x_4]) = 0$. Then there are
integers $i_1$, $i_2$ with $i_1 \leq l \leq j \leq i_2 - 1 \leq m$ such that

\[ x_3 u_{i_1} u_{i_2} x_j \in D \quad \text{and} \quad A(u_2, C[x_3 \cup x_4]) = 0. \]

By (1),

\[ d(u_2, C) = d(u_2, C[x_3, x_j]) = |C[x_3, x_j]| + 1. \]

On the other hand, since $u_2 x_3 \notin D$, using Lemma 2 we obtain that

\[ d^+(u_2, C) = d(u_2, C[x_3, x_j]) + d(u_2, C[x_3, x_i]), \]

\[ \leq |C[x_3, x_i]| + |C[x_3, x_j]| + 1 = |C[x_3, x_j]|, \]

a contradiction.

Now assume that $d^+(u_2, C[x_3 \cup x_4]) = 0$. Let $i$ be minimal as possible, i.e. $d^+(u_2, C[x_3 \cup x_4]) = 0$. Then by

(7) we have $A(u_2, \{x_{m-1}, x_3\}) = \emptyset$. Let $x_i u_j \notin D$, $i \leq j \leq m - 2$ and let $j$ be maximal with these proper-
ties. If $d^+(u_2, C[x_3, x_j]) = 0$, then $d^+(u_2, C) = 0$ because of (6). Since $u_2 x_3$ is a good pair, by the condition of the theorem we have $d^+(u_2, C) + d^+(x_{i+1}) \geq n - 1$. Therefore $d^+(x_{i+1}) \geq n - 2$. On the other hand, it is easy to check that $d^-(x_{i+1}, x_j) = 0$, hence,

\[ d^+(x_{i+1}) \leq n - 3, \]

a contradiction. So we can assume that $d^-(u_2, C[x_3, x_i]) \neq 0$. Let $u_2 x_2 \in D$, where $x_2 \in C[x_2, x_i]$,

and $k$ be minimal as possible. Then, from the minimality of $i$ and $k$ it follows that $A(u_2, C[x_2 \cup x_i]) = \emptyset$. Hence, by Claim 3, $k \geq 5$. By (1) (Claim 1) we have

\[ d(u_2, C) = d(u_2, C[x_2, x_i]) = |C[x_2, x_i]| + 1. \]

On the other hand, since $A(u_2, \{x_{m-1}, x_3\}) = \emptyset$ and $u_2 x_3 \notin D$, using Lemma 2 we obtain that

\[ d(u_2, C) = d(u_2, C[x_3, x_j]) + d(u_2, C[x_3, x_i]), \]

\[ \leq |C[x_3, x_i]| + |C[x_3, x_j]| + 1 = |C[x_3, x_j]|, \]

a contradiction.

**Case 2.** $x_2 u_2 \notin D$ and $x_3 u_2 \notin D$ for every $i \in [3, m]$.

Then $d^-(u_2, C[x_3, x_j]) = 0$.

First assume that there is a $x_i \in C[x_2, x_j]$ such that

\[ u_2 x_i \in D \quad \text{and} \quad A(u_2, C[x_2, x_i]) = \emptyset. \]

By Claim 3,

\[ i \geq 5. \]

Let now $C' := C[x_i, x_1]$ and $C' := C[x_2, x_i]$.

Note that $|C'| + |C'| = m$. If $C' \geq 3$ and $d(u_2, C) = d(u_2, C'') = |C'\rangle + 1$. (11)

Since $x_2 u_2 \notin D$, using Claim 3, (11) and Lemma 2 it is not difficult to obtain

\[ N^-(u_2, C) = \{x_1, x_2, \ldots, x_j\} \]

and

\[ N^+(u_2, C) = \{x_i, x_{i+1}, \ldots, x_m, x_1\}. \]  
(12)
If \( u_{s-1}u_2 \notin D \), then \( u_{s-1} \) and \( u_2 \) are not adjacent and hence, \( \{ u_2, u_{s-1} \} \) is a good pair since \( u_2x_1, u_{s-1}x_1 \in D \). Now from (6), (11) and the condition of the theorem it follows that

\[
d(u_2) = d(u_2, C) + d(u_2, R) \leq n - m - 1 + m - 2 = n - 3
\]

and

\[
n + 2 \leq d(u_{s-1}) = d(u_{s-1}, R) + d(u_{s-1}, C)
\]

\[
\leq n - m - 1 + d(u_{s-1}, C).
\]

Therefore \( d(u_{s-1}, C) \geq m + 3 \), and by Lemma 1 \( u_{s-1} \) has a partner on \( C \), which is a contradiction. So we can assume that this is not the case, i.e. \( u_{s-1}u_2 \notin D \). Then by (12) and the maximality of the cycle \( C \) we conclude that

\[
d^-(x_{j+1}, R) \leq n - m - 1, \quad d^+(x_{j+1}, R) = 0,
\]

and hence,

\[
d(x_{j+1}, R) \leq n - m - 1. \tag{14}
\]

\[
d^-(x_{j-1}, R) \leq n + m - 1, \quad d^+(x_{j-1}, R) = d^-(x_{j-1}, \{ u_3 \}). \tag{15}
\]

If \( d^+(x_{j-1}, \{ u_3 \}) = 1 \), then \( x_{j-1}u_3 \in D \) and \( u_2x_{j+1} \notin D \). Therefore by (12), \( x_{j-1} = x_1 \) and \( x_{j+1} = x_m \). Hence we have \( x_{m+1} \in D \) and the cycle \( x_mx_{m+1} \ldots u_{s-1}x_2 \ldots x_m \) longer than \( C \), which is a contradiction. So we can assume that \( d^+(x_{j-1}, R) = 0 \) and therefore, \( d(x_{j-1}, R) \leq n - m - 1 \). From (15) it follows that

\[
d(x_{j-1}, C') \leq |C'| + 1 \quad \text{or} \quad d(x_{j-1}, C') \leq |C'| + 1.
\]

Assume that \( d(x_{j-1}, C') \leq |C'| + 1 \). Then, since \( \{ u_2, x_{j-1} \} \) is a good pair, by (1), (14) and \( x_{j-3}x_{j-1} \notin D \) we have

\[
2n - 1 \leq d(u_2) + d(x_{j-1}) = d(u_2, R \cup C) + d(x_{j-1}, R) + d(x_{j-1}, C' \cup C) \leq 2n - 2,
\]

a contradiction. Similarly we obtain a contradiction if we assume that \( d(x_{j+1}, C') \leq |C'| + 1 \). Second assume that \( d^+(u_2, C[x_3, x_1]) = 0 \). From \( x_2x_1 \in D \) and \( d^-(x, \{ x_{m-1}, x_m \}) = 0 \) it follows that there is a \( x_j \in C[x_2, x_m] \) such that \( x_j \notin D \) and \( A(u_2, C[x_{j+1}, x_m]) = \emptyset \). Note that \( \{ u_2, x_{j+1} \} \) is a good pair. Then \( d^+(x_{j+1}) \geq n - 2 \) since \( d^+(u_2) = 1 \). From \( d^+(x_{j+1}, R) \leq 1 \) implies that \( d^+(x_{j+1}, C) \geq n - 3 \geq m \), which is impossible. Hence, in all possible cases we reach a contradiction. The proof of the theorem is complete.

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