Broadcasting in Knödel Graph

Hayk Grigoryan and Hovhannes A. Harutyunyan
Department of Computer Science and Software Engineering
Concordia University
Montreal, Quebec, Canada H3G 1M8
Email: h_grig@encs.concordia.ca, haruty@cse.concordia.ca

***Abstract*—The *broadcast time* of a graph G, denoted $b(G)$, is the minimum time necessary to complete the broadcasting in G, i.e. from one of vertices send a message to all other vertices. The Knödel graph $W_{\Delta,n}$ is a regular graph of even order and degree Δ where $2 \leq \Delta \leq \lfloor \log_2 n \rfloor$. The broadcast time of the Knödel graph is known only for $W_{\Delta,2^\Delta}$ and for $W_{\Delta-1,2^{\Delta-1}}$. In this paper we present a tight upper and lower bounds on the broadcast time of the Knödel graph for all even n and $2 \leq \Delta \leq \lfloor \log_2 n \rfloor$. We show that $2\left\lfloor \frac{1}{2}\left\lceil \frac{n-2}{2^\Delta-2} \right\rceil \right\rfloor + 1 \leq b(W_{\Delta,n}) \leq \left\lceil \frac{n-2}{2^\Delta-2} \right\rceil + \Delta - 1$.**

I. Introduction

Broadcasting is the process of distributing a message from a node, called the *originator*, to all other nodes of a communication network. Broadcasting is accomplished by placing series of calls over the communication channels of the network and takes place in discrete time units, sometimes called rounds. Each call involves only two nodes, requires one time unit and each node participates in at most one call per unit of time.

Generally, a network can be modeled as a connected graph $G = (V, E)$, where V is the set of all nodes and E is the set of all communication lines. The *broadcast time* $b(v, G)$ or just $b(v)$ of a vertex v in a connected graph G is defined as the minimum time required to inform all the vertices of G. The broadcast time $b(G)$ of a graph $G = (V, E)$ is defined as $b(G) = max\{b(v)|v \in V\}$.

Since after each time unit the number of informed vertices can at most double, for any graph G on n, vertices $b(G) \geq \lceil \log n \rceil$. A graph G with $b(G) = \lceil \log n \rceil$, is called a *broadcast graph*. A broadcast graph with the minimum possible number of edges is called a *minimum broadcast graph(mbg)*.

The Knödel graph $W_{\Delta,n}$ is a regular graph of even order and degree Δ where $2 \leq \Delta \leq \lfloor \log n \rfloor$ (all logarithms in this paper are base 2, unless otherwise specified). It was introduced by Knödel for $\Delta = \lfloor \log n \rfloor$ and was used in an optimal gossiping algorithm [21]. For smaller Δ, the Knödel graph is defined in [8].

In this paper we study the broadcast time problem in the Knödel graph. The broadcast time of the Knödel graph is known only for $W_{\Delta,2^\Delta}$ and for $W_{\Delta-1,2^{\Delta-1}}$. It is shown that $b(W_{\Delta,2^\Delta}) = \Delta(\Delta \geq 1)$ [23],[5],[21] and that $b(W_{\Delta-1,2^{\Delta-1}}) = \Delta(\Delta \geq 2)$ [20],[3].

The Knödel graph was widely studied as an interconnection network topology and has good properties in terms of broadcasting and gossiping. The Knödel graph $W_{\Delta,2^\Delta}$ is one of the three non-isomorphic infinite graph families known to be minimum broadcast and gossip graphs (graphs that have the smallest possible broadcast and gossip times and the minimum possible number of edges). The other two families are the well known hypercube [5] and the recursive circulant graph [23]. The Knödel graph $W_{\Delta-1,2^\Delta-2}$ is a minimum broadcast and gossip graph also for $n = 2^\Delta - 2(\Delta \geq 2)$ [20],[3]. One of the advantages of the Knödel graph as a network topology is that it achieves the smallest diameter among known minimum broadcast and gossip graphs for $n = 2^\Delta(\Delta \geq 1)$. All the minimum broadcast graph families — k-dimensional hypercube, $C(4, 2^k)$-recursive circulant graph and $W_{k,2^k}$ Knödel graph — have the same degree k, but have diameters equal to k, $\left\lceil \frac{3k-1}{4} \right\rceil$ and $\left\lceil \frac{k+2}{2} \right\rceil$ respectively. A detailed description of some graph theoretic and communication properties of these three graph families and their comparison can be found in [6].

As shown in [1], the edges of the Knödel graph can be grouped into dimensions which are similar to hypercube dimensions. This allows to use these dimensions in a similar manner as in hypercube for broadcasting and gossiping. Unlike the hypercube, which is defined only for $n = 2^k$, the Knödel graph is defined for any even number of vertices. Properties such as small diameter, vertex transitivity as a Cayley graph [19], high vertex and edge connectivity, dimensionality, embedding properties [6] make the Knödel graph a good candidate as a network topology and good architecture for parallel computing. $W_{\lfloor \log n \rfloor,n}$ guarantees the minimum time for broadcasting and gossiping. So, it is a broadcast and gossip graph [1],[7],[8]. Moreover, $W_{\lfloor \log n \rfloor,n}$ is used to construct sparse broadcast graphs of a bigger size by interconnecting several smaller copies or by adding and deleting vertices [15],[12],[11],[2],[4],[13],[20],[14].

Multiple definitions are known for the Knödel graph. We use the following definition from [8], which explicitly presents the Knödel graph as a bipartite graph.

Definition 1. *The Knödel graph on even number of vertices n and of degree Δ were $2 \leq \Delta \leq \lfloor \log n \rfloor$ is defined as $W_{\Delta,n} = (V, E)$ where*

$$V = \{(i, j) \mid i = 1, 2 \; j = 0, ..., n/2 - 1\},$$

$$E = \{((1, j), (2, (j + 2^k - 1) \bmod (n/2))) \mid j = 1, ..., n/2 \; k = 0, 1, ..., \Delta - 1\}.$$

We say that an edge $((1, j'), (2, j'')) \in E$ is r-dimensional if $j'' = (j' + 2^r - 1) \bmod (n/2)$ where $r = 0, 1, ..., \Delta - 1$. In this case, $(1, j')$ and $(2, j'')$ are called r-dimensional neighbors.

Also, we say that the edge is modular when $j' + 2^r - 1 > n/2$.

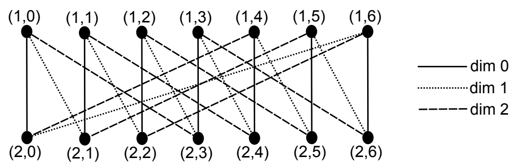

Fig. 1. The $W_{3,14}$ graph and its 0, 1 and 2-dimensional edges

Fig. 1 illustrates $W_{3,14}$ and its 0, 1 and 2-dimensional edges. Despite being a highly symmetric and widely studied graph, the diameter of the Knödel graph $D(W_{\Delta,n})$ is known only for $n = 2^{\Delta}$. In [7], it was proved that $D(W_{\Delta,2^{\Delta}}) = \lceil \frac{\Delta+2}{2} \rceil$. The nontrivial proof of this result is algebraic and the actual diametral path is not presented. In [10] a tight bound is presented for $D(W_{\Delta,n})$ for all even n and $2 \leq \Delta \leq \lfloor \log_2 n \rfloor$. In particular, it is shown that $2 \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor + 1 \leq D(W_{\Delta,n}) \leq 2 \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor + 3$ for almost all Δ. The shortest path problem in $W_{\Delta,2^{\Delta}}$ was studied in [18], where a 2-approximation algorithm with the logarithmic time complexity was presented.

Finding the broadcast time of a given graph is NP-hard [9],[25]. Only for very few graph families a polynomial time algorithm is known. A linear time algorithm for trees is presented in [25]. The algorithm in linear time finds the broadcast center of the tree, and using it, determines the broadcast time for all vertices. In [24], another linear time algorithm for trees is presented. A linear time algorithm is also known for the unicyclic graphs (connected graphs with only one cycle) [16], [17] and few other tree-like graph families [22].

II. Broadcast time of the Knödel graph

In this section we present a tight upper and lower bounds on $b(W_{\Delta,n})$ for all even n and $2 \leq \Delta \leq \lfloor \log_2 n \rfloor$. For the upper bound, we will present a broadcast algorithm in Kn¨odel graph. The lower bound will follow from the known lower bound on the diameter of $W_{\Delta,n}$ from [10].

Let $W'_{\Delta,2^{\Delta}}$ be a graph obtained from $W_{\Delta,2^{\Delta}}$ by removing all the modular edges. See Fig. 2 for an illustration of $W'_{4,16}$. Note that $W'_{\Delta,2^{\Delta}}$ contains only half of edges of the original Knödel graph. The following lemma gives the broadcast time of vertex $(1,0)$ in $W'_{\Delta,2^{\Delta}}$.

Lemma 2. $b((1,0), W'_{\Delta,2^{\Delta}}) = \Delta$.

Proof: It is clear that broadcasting from any originator must take at least Δ time units, since $W'_{\Delta,2^{\Delta}}$ has 2^{Δ} vertices. Therefore, $b((1,\ 0),\ W'_{\Delta,2^{\Delta}}) \geq \Delta$. Next, we present a recursive algorithm for broadcasting in $W'_{\Delta,2^{\Delta}}$ from originator $(1,0)$ in Δ time units. This will prove that $b((1,0), W'_{\Delta,2^{\Delta}}) \leq \Delta$. The recursion will be on Δ.

The base case is when $\Delta = 1$. In this case we have two vertices connected with an edge, therefore $b(W_1{}'_{,2}) = 1$.

For $\Delta > 1$, we note that $W'_{\Delta,2^{\Delta}}$ can be partitioned into two $W'_{\Delta-1,2^{\Delta-1}}$ graphs as illustrated in Fig. 3. The originator $(1,0)$ first will inform its $(\Delta-1)-$dimensional neighbour $(2, 2^{\Delta-1} - 1)$ in $W'_{\Delta,2^{\Delta}}$. After this, both partitions of $W'_{\Delta,2^{\Delta}}$ will have an informed vertex. Each of these two informed vertices will become the new broadcast originator in its $W'_{\Delta-1,2^{\Delta-1}}$ graph. Since at each recursive step we use only one time unit and cut the graph into two equal partitions, it follows that $b(W'_{\Delta,2^{\Delta}}) = \Delta$. ■

Fig. 2 illustrates the broadcast scheme of Lemma 2 in $W'_{4,16}$. The bold edges are used for sending the message and are labeled with the time at which they were used.

Next, we will interpret $W_{\Delta,n}$ as a "chain" of $W'_{\Delta,2^{\Delta}}$ graphs. The idea of the presented broadcast algorithm is to inform one or two special vertices in each of these $W'_{\Delta,2^{\Delta}}$ graphs as soon as it is possible. After getting informed, all these special vertices will start to broadcast in their $W'_{\Delta,2^{\Delta}}$ graphs in parallel as in Lemma 2.

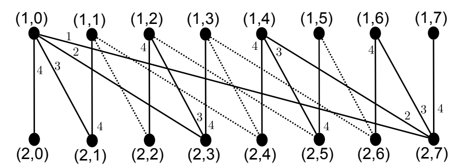

Fig. 2. The $W'_{4,16}$ graph and the broadcast scheme.

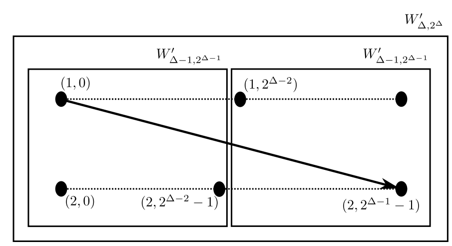

Fig. 3. Recursive partitioning and broadcasting in $W'_{\Delta,2^{\Delta}}$.

Lemma 3 (Theorem 7 in [10]). $D(W_{\Delta,n}) \geq 2 \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor + 1$.

Proof: Note that in order to reach vertex $x = (1, c(2^{\Delta-1} - 1))$ where $c = \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$ from vertex $(2,0)$, we cannot construct a path shorter than illustrated in Fig. 4. This path contains exactly $c+1$ 0-dimensional edges used for changing the graph partition and c $(\Delta-1)$-dimensional edges used for moving towards x in the fastest possible way. Thus, $D(W_{\Delta,n}) \geq 2c + 1 = 2 \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor + 1$. ■

Theorem 4 (Broadcast time). $2 \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor + 1 \leq b(W_{\Delta,n}) \leq \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil + \Delta - 1$.

Proof: The lower bound follows from the lower bound on $D(W_{\Delta,n})$ from Lemma 3), since obviously we will need at least $D(W_{\Delta,n})$ time units to inform a vertex at distance $D(W_{\Delta,n})$ from the broadcast originator.

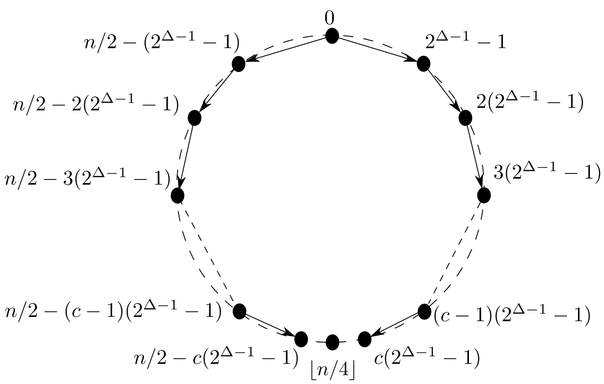

Fig. 4. Schematic illustration of the paths. $c = \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$

To prove the upper bound, we present an algorithm for broadcasting in $W_{\Delta,n}$. The algorithm uses at most $\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil + \Delta - 1$ time units.

By considering only 0 and $(\Delta-1)$-dimensional edges, the Knödel graph can be schematically illustrated as in Fig. 4. Recall that the $\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil$ expression represents the number of partitions in Fig. 4. We note that each partition is a $W'_{\Delta,2^{\Delta}}$ graph. More precisely, we have $\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil - 1$ partitions of the form $W'_{\Delta,2^{\Delta}}$ and one partition of the form $W'_{\Delta,n-(\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil-1)(2^{\Delta-1}-1)-1}$.

The broadcast algorithm for $W_{\Delta,n}$ consists of three stages. In the first stage, we inform all the vertices with labels $(1,0)$ and $(2, 2^{\Delta-1}-1)$ in all W' graphs except one or two farthest W' graphs from the originator. In the second stage, we use Lemma 2 to broadcast in parallel in all W' graphs. In the third stage, all the vertices of the remaining one or two W' graphs will receive the message in just 1 or 2 time units from neighboring W' graphs and the broadcast will be complete in $W_{\Delta,n}$.

We note that the vertices of $W_{\Delta,n}$ with original labels $y = (1, c(2^{\Delta-1}-1))$ and $y = (2, n/2 - c(2^{\Delta-1}-1))$ where $0 \leq c \leq \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$ after relabeling become the vertices with label $(1,0)$ in W' partitions. Similarly, vertices $y = (2, c(2^{\Delta-1}-1))$ and $y = (1, n/2 - c(2^{\Delta-1}-1))$ where $0 \leq c \leq \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$ become the vertices $(2, 2^{\Delta-1}-1)$ in W' graphs. Therefore, we can use the paths from Fig. 4 in the first stage of the broadcasting. All the vertices which need to be informed in the first stage form a "cycle" of length $\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil$ in $W_{\Delta,n}$. Each "edge" of this cycle consists of one 0 and one $(\Delta-1)$-dimensional edge and it takes 2 time units to send a message via such edge. It follows that we need $2 \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$ to complete the first stage of broadcasting, i.e inform all the vertices of this "cycle" except one or two farthest ones from the originator $(1,0)$. In order to have a goodupper bound on $b(W_{\Delta,n})$, we consider the parity of $\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil$.

If $\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil$ is odd when it will take $2 \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor - 1 = \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil - 2$ rounds to complete the first stage. After this, all the W' partitions of the Knödel graph, except two farthest ones from the originator $(1,0)$, will have their vertices with label $(1,0)$ and $(2, 2^{\Delta-1}-1)$ informed. We note that by the end of the first stage, the first step of recursive broadcast algorithm from Lemma 4 will be complete. This means that we need only $\Delta-1$ additional rounds to inform all the vertices in W' graphs. Finally, in the third stage, in just 2 time units the final two uninformed W' graphs will receive the broadcast message from the neighboring and fully informed W'. At first, the $(\Delta-1)$-dimensional edges will be used to inform all vertices in one of the partitions in the reaming 2 W' graphs. After this, the 0-dimensional edges will be used to inform all the vertices of the second partition. It follows that $b(W_{\Delta,n}) \leq (\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil - 2) + (\Delta-1) + 2 = \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil + \Delta - 1$.

If $\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil$ is even when it take $\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil - 1$ rounds to complete the first s tage. W e n ote t hat i n t his c ase a ll W' graphs except one, will have two vertices with labels $(1,0)$ and $(2, 2^{\Delta-1}-1)$ informed. As in the previous case, we will need only $\Delta-1$ time units to complete the broadcasting in W' graphs according to Lemma 4. In the third stage, in just one time unit, using $(\Delta-1)$-dimensional edges we will inform all the vertices of the remaining W' graph from neighboring W' graphs. Hence, in this case we have $b(W_{\Delta,n}) \leq (\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil - 1) + (\Delta-1) + 1 = \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil + \Delta - 1$ as well. ■

Fig. 5 illustrates the broadcast algorithm of Theorem 4 in $W_{3,32}$ graph. For this case the number of partitions $\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil = \left\lceil \frac{30}{6} \right\rceil = 5$ is odd and we deal with the first case of Theorem 4. The 0 and 2-dimensional edges divide the $W_{3,32}$ graph into 5 parts $S_1, S_2, ..., S_5$. Each part is a $W'_{3,8}$ graph. The goal of the first stage of the broadcast algorithm is to inform two special vertices in S_1, S_2 and S_5 partitions. These are the vertices $(1,0)$ and $(2,3)$ in S_1, $(1,3)$ and $(2,6)$ in S_2, $(2,0)$ and $(1,13)$ in S_5. The bold edges are used to accomplish this in 3 time units. After relabeling, these special vertices are going to have labels $(1,0)$ and $(2,3)$ in $W'_{3,8}$ partitions. During the second stage of the broadcasting, all these vertices will broadcast in parallel in S_1, S_2 and S_5 partitions as shown in Fig. 6. From Lemma 2 follows that we need only 2 time units to broadcast from originators $(1,0)$ and $(2,3)$ in $W'_{3,8}$ i.e. $b(\{(1,0),(2,3)\}, W'_{3,8}) = 2$. The broadcast scheme is illustrated in Fig. 5. It follows that the second stage will be complete in 2 time units. Finally, in 2 more time units, the vertices of S_2 and S_5 will inform all the vertices of S_3 and S_4. The total broadcast time will be $b(W_{3,32}) \leq 3+2+2 = 7$.

Fig. 7 illustrates the broadcast algorithm of Theorem 4 in $W_{3,26}$ graph. For this case the number of partitions $\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil = \left\lceil \frac{24}{6} \right\rceil = 4$ is even and we deal with the second case of Theorem 4. The 0 and 2-dimensional edges divide the $W_{3,32}$ graph into 4 parts S_1, S_2, S_3, S_4. Each part is a $W'_{3,8}$ graph. For this case, the goal of the first stage of the broadcast algorithm is to inform two special vertices in S_1, S_2 and S_4 partitions. The bold edges are used to accomplish this in 3 time units. As in the case of $W_{3,32}$, from Lemma 2 follows that we need only 2

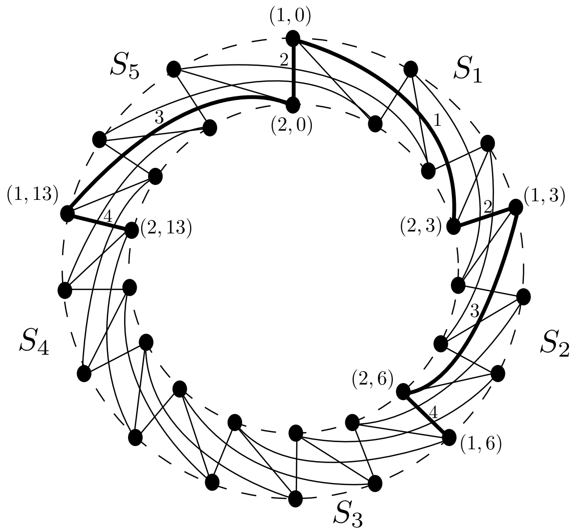

Fig. 5. Broadcast scheme in $W_{3,32}$.

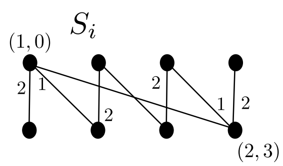

Fig. 6. Two originator broadcast scheme in $W'_{3,8}$.

time units to inform all vertices in S_1, S_2 and S_4 (see Fig. 6). The broadcast scheme is illustrated in Fig. 7. Finally, in just 1 time unit, the vertices of S_2 and S_4, using 2-dimensional edges, will inform all the vertices of S_3. The total broadcast time will be $b(W_{3,26}) \leq 3 + 2 + 1 = 6$.

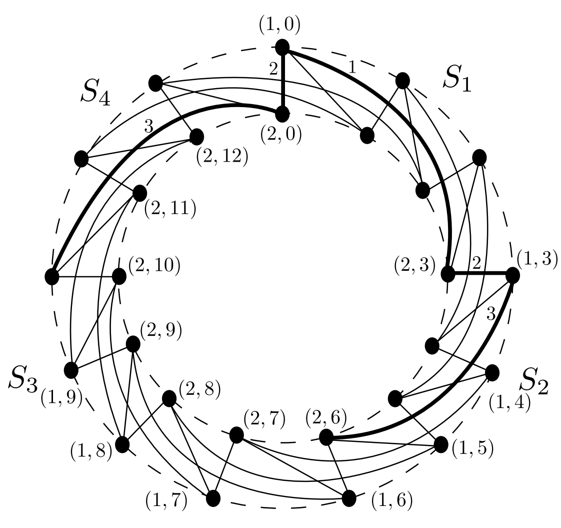

Fig. 7. Broadcast scheme in $W_{3,26}$.

III. Summary

For the broadcast time of the Knödel graph $W_{\Delta,n}$, we showed that $2\left\lfloor \frac{1}{2}\left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor + 1 \leq b(W_{\Delta,n}) \leq \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil + \Delta - 1$. We believe that the presented lower bound, based only on $D(W_{\Delta,n})$, can be improved. In fact, we state as a conjecture that $b(W_{\Delta,n}) = \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil + \Delta - 1$.

References

[1] J. C. Bermond, H. A. Harutyunyan, A. L. Liestman, and S. Perennes. A note on the dimensionality of modified knödel graphs. *Int. J. Found. Comput. Sci.*, 8(2):109–116, 1997.

[2] J.C. Bermond, P. Fraigniaud, and J.G. Peters. Antepenultimate broadcasting. *Networks*, 26(3):125–137, 1995.

[3] M. Dinneen, M. Fellows, and V. Faber. Algebraic constructions of efficient broadcast networks. *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, pages 152–158, 1991.

[4] Michael J Dinneen, Jose A Ventura, Mark C Wilson, and Golbon Zakeri. Compound constructions of broadcast networks. *Discrete applied mathematics*, 93(2):205–232, 1999.

[5] A.M. Farley, S. Hedetniemi, S. Mitchell, and A. Proskurowski. Minimum broadcast graphs. *Discrete Mathematics*, 25:189–193, 1979.

[6] G. Fertin and A. Raspaud. A survey on knödel graphs. *Discrete applied mathematics*, 137(2):173–195, 2004.

[7] G. Fertin, A. Raspaud, H. Schröder, O. Sỳkora, and I. Vrt'o. Diameter of the knödel graph. In *Graph-Theoretic Concepts in Computer Science*, pages 169–191. Springer, 2000.

[8] P. Fraigniaud and J. G. Peters. Minimum linear gossip graphs and maximal linear (δ, k)-gossip graphs. *Networks*, 38(3):150–162, 2001.

[9] M.R. Garey and D.S. Johnson. *Computers and intractability*. Freeman San Francisco, 1979.

[10] H. Grigoryan and H.A. Harutyunyan. Tight bound on the diameter of the knödel graph. In *International Workshop On Combinatorial Algorithms (to appear)*, 2013.

[11] H.A. Harutyunyan. Minimum multiple message broadcast graphs. *Networks*, 47(4):218–224, 2006.

[12] H.A. Harutyunyan. An Efficient Vertex Addition Method for Broadcast Networks. *Internet Mathematics*, 5(3):211–225, 2008.

[13] H.A. Harutyunyan and A.L. Liestman. More broadcast graphs. *Discrete Applied Mathematics*, 98(1-2):81–102, 1999.

[14] H.A. Harutyunyan and A.L. Liestman. On the monotonicity of the broadcast function. *Discrete Mathematics*, 262(1-3):149–157, 2003.

[15] H.A. Harutyunyan and A.L. Liestman. Upper bounds on the broadcast function using minimum dominating sets. *Discrete Mathematics*, 312(20):2992–2996, 2012.

[16] H.A. Harutyunyan and E. Maraachlian. Linear algorithm for broadcasting in unicyclic graphs. In *International Computing and Combinatorics Conference*, pages 372–382, 2007.

[17] H.A. Harutyunyan and E. Maraachlian. On broadcasting in unicyclic graphs. *Journal of Combinatorial Optimization*, 16(3):307–322, 2008.

[18] H.A. Harutyunyan and C.D. Morosan. On the minimum path problem in knödel graphs. *Networks*, 50(1):86–91, 2007.

[19] M. C. Heydemann, N. Marlin, and S. Pérennes. Complete rotations in cayley graphs. *European Journal of Combinatorics*, 22(2):179–196, 2001.

[20] L.H. Khachatrian and O.S. Harutounian. Construction of new classes of minimal broadcast networks. In *Conference on Coding Theory, Dilijan, Armenia*, pages 69–77, 1990.

[21] W. Knödel. New gossips and telephones. *Discrete Mathematics*, 13(1):95, 1975.

[22] E. Maraachlian. *Optimal broadcasting in treelike graphs*. PhD thesis, Montreal, P.Q., Canada, 2010.

[23] J.H. Park and K.Y. Chwa. Recursive circulant: a new topology for multicomputer networks (extended abstract). In *Parallel Architectures, Algorithms and Networks, 1994.(ISPAN), International Symposium on*, pages 73–80, 1994.

[24] A. Proskurowski. Minimum broadcast trees. *IEEE transactions on computers*, 100(30):363–366, 1981.

[25] P.J. Slater, E.J. Cockayne, and S.T. Hedetniemi. Information dissemination in trees. *SIAM Journal on Computing*, 10:692–701, 1981.