Indivisible Lines in the Space of Discrete Geometries

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ABSTRACT
Existence of 'indivisible lines' in the space of discrete geometries is proved.

Keywords
Discrete space, discrete geometry, indivisible lines, prime number.

1. INTRODUCTION
An important problem of discrete mathematics is the question of existence of 'indivisible lines' [1]. The pair of points, (x, y), which are the ends of the 'indivisible line' and which contain nothing between them was considered. The recent result [2] about the existence of a kernel in the space of discrete geometries [3] hinted the exis-tence of indivisible lines in such spaces. It was found that the existence of indivisible lines is connected with the problem of the infinite sequence of prime numbers. Be-low the main ideas of the above-said considerations are presented in details. But first, some auxiliary notions from [3–5] are presented.

2. AUXILIARY RESULTS
Definition 1. *The set of real numbers,* $\mathcal{R}$ $(|\mathcal{R}| \geqslant 3)$ *without zero, is called scalar product if it satisfies the following two conditions:*
a) $A + B > 0$*, for any pair,* $A, B \in \mathcal{R}$ *if they are different;*
b) $AB+AC+BC \geqslant 0$*, for any number triplet,* $A, B, C \in \mathcal{R}$.

Theorem 1. *To any real number triplet,* $A, B, C \in \mathcal{R}$*, can be assigned a triangle* $\triangle A_1B_1C_1$ *with the sides having the lengths:*

$$a = |B_1C_1|, \quad b = |A_1C_1|, \quad c = |A_1B_1|, \tag{1}$$

where

$$\begin{aligned} A &= \frac{b^2 + c^2 - a^2}{2}, \\ B &= \frac{a^2 + c^2 - b^2}{2}, \\ C &= \frac{a^2 + b^2 - c^2}{2}. \end{aligned} \tag{2}$$

Inversely, any Euclidean triangle,$\triangle A_1B_1C_1$, can be associated with an equal to it Euclidean triangle with the number denotations, $\triangle ABC$, where:

$$a = \sqrt{B + C}, \quad b = \sqrt{A + C}, \quad c = \sqrt{A + B}. \tag{3}$$

Taking (3) Heron's formula into account, the area of the triangle $\triangle ABC$, takes the form:

$$S = \frac{1}{2}\sqrt{AB + AC + BC} \geqslant 0. \tag{4}$$

According to (3) and (4), we obtain the formulae of the discrete trigonometric functions:

$$\begin{aligned} &\sin\angle ABC = \frac{2S}{\sqrt{B^2 + 4S^2}}, \\ &\cos\angle ABC = \frac{B}{\sqrt{B^2 + 4S^2}}, \\ &\mathrm{tg}\angle ABC = \frac{2S}{B} \\ &B = 2S \cdot \mathrm{ctg}\angle ABC. \end{aligned} \tag{5}$$

It was shown in [2–4] that for any fixed integer $D \in (-\infty, -2]$ the infinite and asymmetric set of integers:

$$\begin{aligned} &\mathcal{N}(D) = \\ &= \{D, -D + 1\} \cup \left\{D^2 - D + i,\ i = 0, 1, 2, \ldots\right\} \end{aligned} \tag{6}$$

forms a discrete and infinite metric space with the metric (7):

$$\begin{aligned} &r(X, Y) = \begin{cases} \sqrt{X + Y}, & X \neq Y \\ 0, & X = Y \end{cases} \\ &X, Y \in \mathcal{N}(D). \end{aligned} \tag{7}$$

One can easily be convinced that the space $\mathcal{N}(D)$ is contracted if $D \to -\infty$. In the space $\mathcal{N}(D)$ for the fixed $D_0 \leqslant -2$, three discrete objects are introduced $\mathcal{E}$, $\overline{\mathcal{E}}$, $\mathcal{P}$ together with their defining points: $\mathcal{E}\{D_0, K, A, B\}$, $\overline{\mathcal{E}}\{D_0, K, A, B, C\}$, $\mathcal{P}\{D_0, K, A, B, C, E, F\}$, which satisfy the system of equations (8):

$$\left.\left.\left.\begin{aligned} D_0 + K = 1 \\ A + D_0 = B + K \\ AD_0 + BK = 0 \end{aligned}\right] \mathcal{E} \atop KC + D_0C + KD_0 = 0 \right] \overline{\mathcal{E}} \atop \begin{aligned} D_0^2 + K^2 + C^2 = E^2 \\ A^2 + B^2 + C^2 = F^2 \end{aligned}\right] \mathcal{P} \tag{8}$$
$$\mathcal{E} \supset \overline{\mathcal{E}} \supset \mathcal{P}.$$

For the fixed $D_0 \leqslant -2$ each of the system $\mathcal{E}$, $\overline{\mathcal{E}}$, $\mathcal{P}$, has its unique solution in integers (9) expressed in the odd parameter $n = 1 - 2D_0$ $(n \geqslant 5)$:

$$\begin{aligned} D_0 = -\frac{n-1}{2},\ \ K = \frac{n+1}{2},\ \ C = \frac{n^2-1}{4}, \\ E = \frac{n^2+3}{4},\ \ B = \frac{n(n-1)}{2}, \\ A = \frac{n(n+1)}{2},\quad F = \frac{3n^2+1}{4}, \end{aligned} \tag{9}$$

3. THE MAIN RESULT
Definition 2. *For any fixed* $D = D_0 \in (-\infty, -2]$ *the infinite set* $\mathcal{N}(D_0)$ (6) *is called* D_0*-cut of the sub-space* $\mathcal{N}(D)$ *and is denoted by* $\mathcal{N}(D_0)$.

Definition 3. The pair of the points $(D_0, |D_0|+1) = (D_0, K_0) \in \mathcal{N}(D_0)$*, satisfying the following two conditions:*

$$D_0 + K_0 = 1 \qquad (10)$$

$$D_0^2 + K_0^2 = p_0 \qquad (11)$$

is called an indivisible line of D_0*-cut* $\mathcal{N}(D_0)$*, where* p_0 *– is a prime number of the form* $60m+1$*.*

It follows from (10) that for any positive odd number $p_0 = 1 - 2D_0 \geqslant 5$, the following holds:

$$D_0 = -\frac{p_0 - 1}{2}, \quad K_0 = \frac{p_0 + 1}{2}. \qquad (12)$$

According to (11) and (12) we build a recurrent formula for the sequence $\{p_k\}$, where

$$p_k = \left(\frac{p_{k-1} - 1}{2}\right)^2 + \left(\frac{p_{k-1} + 1}{2}\right)^2 = \frac{p_{k-1}^2 + 1}{2}. \qquad (13)$$

In (13) we take $p_0 = 61$ and calculate the first three terms of the sequence $\{p_k\}$ $k = 1, 2, 3, \ldots$. We get:

$$p_1 = \left(\frac{p_0 - 1}{2}\right)^2 + \left(\frac{p_0 + 1}{2}\right)^2 =$$

$$= 30^2 + 31^2 = 1861 = 60 \cdot 31 + 1 = 60\frac{p_0 + 1}{2} + 1$$

$$p_2 = \left(\frac{p_1 - 1}{2}\right)^2 + \left(\frac{p_1 + 1}{2}\right)^2 =$$

$$= 930^2 + 931^2 = 1731661 = 60 \cdot 31 \cdot 931 + 1 =$$

$$= 60\frac{p_0 + 1}{2} \cdot \frac{p_1 + 1}{2} + 1$$

$$p_3 = 865830^2 + 865831^2 = 1499324909461 =$$

$$= 60\frac{p_0 + 1}{2} \cdot \frac{p_1 + 1}{2} \cdot \frac{p_2 + 1}{2} + 1$$

$$\ldots \ldots \ldots$$

$$p_k = \left(\frac{p_{k-1} - 1}{2}\right)^2 + \left(\frac{p_{k-1} + 1}{2}\right)^2 =$$

$$= 60\prod_{i=0}^{k-1} \frac{p_i + 1}{2} + 1 = \left(60\prod_{i=0}^{k-2} \frac{p_i + 1}{2}\right) \times$$

$$\times \frac{p_{k-1} + 1}{2} + 1 = A\frac{p_{k-1} + 1}{2} + 1. \qquad (14)$$

According to (14) the value A has the form:

$$A = 60\prod_{i=0}^{k-2} \frac{p_i + 1}{2}. \qquad (15)$$

According to (13), (14) and (15) we obtain an equation for p_{k-1}:

$$A\frac{p_{k-1} + 1}{2} + 1 = \frac{p_{k-1}^2 + 1}{2}$$

or

$$p^2{}_{k-1} - Ap_{k-1} - A - 1 = 0. \qquad (16)$$

Solving the equation (16), we obtain the term p_{k-1} in the sequence $\{p_k\}$:

$$p_{k-1} = A + 1. \qquad (17)$$

according to (15) and (17), we have:

$$p_{k-1} = 60\prod_{i=0}^{k-2} \frac{p_i + 1}{2} + 1. \qquad (18)$$

Using Euler's formula $x^2 + y^2 = N$, and choosing an appropriate form for [6], it was shown that the obtained numbers, p_1, p_2, p_3 are prime. Taking into account that the numbers p_0, p_1, p_2, p_3 – are prime and the fact that the sequence $\{p_i\}$, has direct (14) and inverse (18) recurrences, it follows that this sequence is composed of infinite number of prime numbers of the form $p_i = 60m_i + 1$. According to Definition 3, this provides the existence of the infinite number of indivisible lines in the discrete metric space $\mathcal{N}(D)$ (6).

The indivisible lines together with the prime numbers can have general geometrical and arithmetical significance and content in the space, $\mathcal{N}(D)$ (6), due to their unique complex of properties and peculiarities, viz. due to their discreteness, infinity, a symmetry, non-homogeneousness, null-dimensions, quantized character, etc.

REFERENCES

[1] A.N. Vyaltsev, "Discrete Space-Time", *Moscow, ISBN 978-5-484-00866-7, ComKniga, (in Russian)*, 2007.

[2] Yu.G. Grigoryan, "Asymmetry and Kernel of the Coded Metric Space", *J. Mathematical Problems of Computer Science, (in Russian)*, XXXVIII, Yerevan, pp. 10-11, 2012.

[3] Yu.G. Grigoryan, "Principles of Inhomogeneous Geometry", *J. Algebra, Geometry, & Their Applications, seminar proceedings, ISN5-8084-0352-9AGA*, Vol. 3-4, p. 40-53, 2004.

[4] Yu.G. Grigoryan, "Discrete Geometries' Space", *J. Cybernetics and Systems Analyses*, Vol. 42, N 5, p. 631-640, 2006.

[5] Yu.G. Grigoryan, "Nonclassical Properties of a Discrete Geometries Space", *J. Cybernetics and Systems Analyses*, Vol. 45, N 5, p. 714-722, 2009.

[6] A.A. Buhshtab, "Number Theory", *Moscow, "Prosveshchenie", (in Russian)*, 1966.