On interval edge-colorings of complete tripartite graphs

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ABSTRACT

An edge-coloring of a graph G with colors $1, \dots, t$ is an interval t-coloring if all colors are used, and the colors of edges incident to each vertex of G are distinct and form an interval of integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. In this paper we prove that $K_{1,m,n}$ is interval colorable if and only if $\gcd(m+1, n+1) = 1$, where $\gcd(m+1, n+1)$ is the greatest common divisor of $m+1$ and $n+1$.

Keywords

Edge-coloring, interval edge-coloring, complete multipartite graph, complete tripartite graph

1. INTRODUCTION

All graphs in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$, the maximum degree of G by $\Delta(G)$ and the edge-chromatic number of G by $\chi'(G)$. The terms, notations and concepts that we do not define can be found in [16].

A proper edge-coloring of graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. If α is a proper coloring of G and $v \in V(G)$, then $S(v, \alpha)$ (spectrum of a vertex v) denotes the set of colors of edges incident to v. A proper edge-coloring of a graph G with colors $1, \dots, t$ is an interval t-coloring if all colors are used, and for any vertex v of G, the set $S(v, \alpha)$ is an interval of integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the least and the greatest values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively.

The concept of interval edge-coloring was introduced by Asratian and Kamalian [3]. In [3, 4], they proved that if G is interval colorable, then $\chi'(G) = \Delta(G)$. Moreover, they also showed that if G is a triangle-free graph and $G \in \mathfrak{N}$, then $W(G) \leq |V(G)| - 1$. In [8], Kamalian investigated interval edge-colorings of complete bipartite graphs and trees. Later, Kamalian [9] showed that if G is a connected graph and $G \in \mathfrak{N}$, then $W(G) \leq 2|V(G)| - 3$. This upper bound was improved by Giaro, Kubale, Malafiejski in [7], where they proved that if G ($|V(G)| \geq 3$) is a connected graph and $G \in \mathfrak{N}$, then $W(G) \leq 2|V(G)| - 4$. Recently, Kamalian and Petrosyan [10] showed that if G was a connected r-regular graph ($|V(G)| \geq 2r + 2$) and $G \in \mathfrak{N}$, then $W(G) \leq 2|V(G)| - 5$. Interval edge-colorings of planar graphs were considered by Axenovich in [5], where she proved that if G is a connected planar graph and $G \in \mathfrak{N}$, then $W(G) \leq \frac{11}{6}|V(G)|$. In [12], Petrosyan investigated interval colorings of complete graphs and n-dimensional cubes. In particular, he proved that if $n \leq t \leq \frac{n(n+1)}{2}$, then the n-dimensional cube Q_n has an interval t-coloring. Recently, Petrosyan, Khachatrian and Tananyan [14] showed that the n-dimensional cube Q_n has an interval t-coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$. In [15], Sevast'janov proved that it is an NP-complete problem to decide whether a bipartite graph has an interval coloring or not.

Interval edge-colorings of some special cases of complete multipartite graphs were first considered by Kamalian in [8], where he proved the following theorem.

Theorem 1. For any $m, n \in \mathbb{N}$, $K_{m,n} \in \mathfrak{N}$ and

1 $w(K_{m,n}) = m + n - \gcd(m, n)$

2 $W(K_{m,n}) = m + n - 1$

3 *if $w(K_{m,n}) \leq t \leq W(K_{m,n})$, then $K_{m,n}$ has an interval t-coloring.*

Also, he showed that complete graphs were interval col-orable if and only if the number of vertices was even.
Moreover, for any $n \in \mathbb{N}$, $w(K_{2n}) = 2n - 1$. For a lower bound on $W(K_{2n})$, Kamalian obtained the following result:

Theorem 2. For any $n \in \mathbb{N}$, $W(K_{2n}) \geq 2n - 1 + \lfloor \log_2 (2n - 1) \rfloor$.

Later, Petrosyan [12] improved this lower bound for $W(K_{2n})$:

Theorem 3. If $n = p2^q$, where p is odd and q is non-negative, then

$$W(K_{2n}) \geq 4n - 2 - p - q.$$

In the same paper he also conjectured that this lower bound was the exact value of $W(K_{2n})$. He verified this conjecture for $n \leq 4$, but the conjecture was disproved by the second author in [11].

Another special case of complete multipartite graphs was considered by Feng and Huang in [6], where they proved the following:

Theorem 4. For any $n \in \mathbb{N}$, $K_{1,1,n} \in \mathfrak{N}$ if and only if n is even.

Recently, Petrosyan investigated interval edge-colorings of complete multipartite graphs. In particular, he proved [13] the following result:

Theorem 5. If $K_{n,\dots,n}$ is a complete balanced k-partite graph, then $K_{n,\dots,n} \in \mathfrak{N}$ if and only if nk is even. Moreover, if nk is even, then $w(K_{n,\dots,n}) = n(k-1)$ and $W(K_{n,\dots,n}) \geq \left(\frac{3}{2}k - 1\right)n - 1$.

In "Cycles and Colorings 212" workshop Petrosyan presented several conjectures on interval edge-colorings of complete multipartite graphs. In particular, he posed the following:

Conjecture 1. For any $m, n \in \mathbb{N}$, $K_{1,m,n} \in \mathfrak{N}$ if and only if $\gcd(m+1, n+1) = 1$.

In this paper we prove this conjecture, which also generalizes Theorem 4.

2. MAIN RESULTS

Theorem 6. If $\gcd(m+1, n+1) = 1$, then $K_{1,m,n}$ has an interval $(m+n)$-coloring.

To prove this result we color the subgraph isomorphic to $K_{m,n}$ using $m+n-1$ colors. Next we construct an auxillary graph $H_{m,n}$ and show that if $H_{m,n}$ has a perfect matching, then it is possible to color the remaining edges of $K_{1,m,n}$ and obtain an interval edge-coloring. Finally, we show that $H_{m,n}$ has a perfect matching if $m+1$ and $n+1$ are coprime.

Theorem 7. If $\gcd(m+1, n+1) = 1$ and $t > m+n+1$ then $K_{1,m,n}$ does not have an interval t-coloring.

In the proof of the theorem we show that if there exist two edges with colors 1 and t ($t \geq m+n+2$) then there exists a vertex with a non-interval spectrum. This proves the theorem.

Theorem 8. If $\gcd(m+1, n+1) > 1$ then $K_{1,m,n}$ is not interval colorable.

We prove the theorem by contradiction. We suppose that $\gcd(m+1, n+1) = d > 1$ and $K_{1,m,n}$ has an interval edge-coloring. For each vertex $w \in V(K_{1,m,n})$ we count the number of edges with color dx (for some $x \in \mathbb{Z}$) incident to w. The sum of these numbers yields an odd number. This contradiction proves the theorem.

The question whether $K_{1,m,n}$ has an interval $(m+n+1)$-coloring when $\gcd(m+1, n+1) = 1$, is still open.

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