New methods of construction of fault-tolerant gossip graphs *

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ABSTRACT

The gossip problem (telephone problem) is an information dissemination problem in which each of n nodes of communication network has a unique piece of information that should be transmitted to all theother nodes us- ing two-way communications (telephone calls) between the pairs of nodes. During a call between the given two nodes, they exchange the whole information known to them at that moment. In this paper we investigate the k-fault-tolerant gossip problem, which is a generaliza- tion of the gossip problem, where at most k arbitrary faults of calls are allowed. The problem is to find the minimal number of calls τ $(n,\ k)$ required to ensure the k-fault-tolerance. We construct two classes of k-fault-tolerant gossip schemes (sequences of calls) and improve the previously known results on upper bounds of τ $(n,\ k)$. Assuming that calls can be made simultaneously, it is also of interest to find k-fault-tolerant gossip schemes, which can spread the full information during a minimal time. For even n we showed that the minimal time is $T(n,k) = \lceil \log_2 n \rceil + k$.

Keywords

Networks, telephone problem, gossip problem, fault-tolerant communication.

1. INTRODUCTION

It has been shown in numerous works (see ex. [1]) that the minimal number of calls in gossip problem is $2n-4$ when $n \geq 4$ and $0, 1, 3$ for $n = 1, 2, 3$, respectively. Since then many variations of gossip problem have been introduced and investigated (see ex. [2]).

One of the natural generalizations of gossip problem is the k-fault-tolerant gossip problem, which assumes that some of the calls in the call sequence can fail (not occur) [4, 5, 6, 7]. The nodes cannot change the sequence of their future calls depending on the current failed calls. Here the aim is to find a minimal gossip scheme, which guarantees the full exchange of information in the case of at most k arbitrary fails, regardless of which calls are failed. The gossip schemes, which provide k-fault-tolerance, are called k-fault-tolerant gossip schemes. Denote the minimal number of calls in the k-fault-tolerant minimal gossip scheme by $\tau(n,k)$.

*The main results that we obtained are presented in [8]. Here is a new method of constructing and indirectly introducing Knödel method of construction.

Berman and Hawrylycz [5] obtained the lower and upper bounds for $\tau(n,k)$:

$$\tau(n,k) \leq \left\lfloor \left(k + \frac{3}{2}\right)(n-1) \right\rfloor \tag{1.1}$$

for $k \leq n-2$, and

$$\tau(n,k) \leq \left\lfloor \left(k + \frac{3}{2}\right)(n-1) \right\rfloor \tag{1.2}$$

for $k \geq n-2$.

For the special case when $n = 2^p$ for some integer p, Haddad, Roy, and Schaffer [6] showed that $\tau(n,k) \leq \frac{nk}{2} + O(n \log_2 n)$.

Later on, Hou and Shigeno [4] showed that $\frac{nk}{2} + \Omega(n) \leq \tau(n,k) \leq \frac{nk}{2} + O(n^2)$. These bounds improve the previous bounds for small n and sufficiently large k.

Recently, Hasunuma and Nagamochi [7] showed that $\tau(n,k) \leq \frac{nk}{2} + O(n \log_2 n)$. Particularly, their upper bound improves the upper bound by Hou and Shigeno for all $n \geq 13$.

Our main methods of construction are presented in [8] and here are the main results

$$\xi_1(n,k) = \begin{cases} \frac{n\lceil \log_2 n\rceil}{2} + \frac{nk}{2}, & \text{for } n \text{ even} \\ \frac{(n-1)\lceil \log_2 (n-1)\rceil}{2} + \frac{(n-1)k}{2} + 2(k+1), & \text{for } n \text{ odd,} \end{cases} \tag{1.3}$$

but in this paper we construct k-fault tolerant gossip schemes based on graphs combination method, with number of calls

$$\xi_2(n,k) = \begin{cases} (k+1)(n-2)+3, & \text{for } n \geq 6, \quad k \text{ even} \\ (k+1)(n-2)+4, & \text{for } n \geq 6, \quad k \text{ odd}, \quad k \neq 1 \end{cases} \tag{1.4}$$

respectively. Therefore, the number of calls of a k-fault-tolerant minimal gossip scheme has an upper bound

$$\tau(n,k) \leq \min\left\{\xi_1(n,k), \xi_2(n,k)\right\}, \tag{1.5}$$

which holds for general $n \geq 6$ and $k \geq 1$, and improves the above mentioned previously known results.

Assuming that the calls can occur simultaneously, it is also of interest to find gossip schemes, which can spread the full information during a minimal time. For even n we showed([8]) that the minimal time is $T(n,k) = \lceil \log_2 n \rceil + k$.

2. K-FAULT-TOLERANT GOSSIP GRAPHS BASED ON KNÖDEL GRAPHS

The family of Knödel graphs ([3]) is defined as follows:

Definition 1. *The Knödel graph on $n \geq 2$ vertices (n even) and of maximum degree $\Delta \geq 1$ is denoted $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the pairs (i, j) with $i = 1, 2$ and $0 \leq j \leq n/2 - 1$. For every j, $0 \leq j \leq n/2 - 1$ and $l = 1, \dots, \Delta$, there is an edge with the label l between the vertex $(1, j)$ and $(2, (j + 2^{l-1} - 1) \bmod n/2)$. The edges with the given label l are said to be in dimension l.*

Note that $W_{1,n}$ consists of $n/2$ disconnected edges. For $\Delta \geq 2$, $W_{\Delta,n}$ is connected.

For constructing a k-fault tolerant graph we suggest to use the union of two Knödel graphs and increase the value of Δ_2. By incrementing it we also increment the value of k until we reach $\Delta_2 = \lfloor \log_2 n \rfloor$. Then, we suggest to involve a new $W_{\Delta,n}$ graph in G and repeat the actions described above.

Let τ (n, k) denote the minimum number of edges in a k-fault tolerant gossip graph with n vertices.

Theorem 1. *In k fault-tolerant gossip schemes the upper bound of $\tau(n, k)$ minimum needed calls in case when n is even($n \neq 2^p$) satisfies the following condition:*

$$\tau(n, k) \leq \frac{n}{2} \lceil \log_2 n \rceil + \frac{nk}{2}. \tag{2.1}$$

This theorem is formulated and proved in [8].

Note that for even n this method of construction provides the minimum time for fault-tolerant gossiping - $T(n, k) = \lceil \log_2 n \rceil + k$.

In case when n is odd we have to make incoming and outgoing calls as much as necessary to meet the fault-tolerant gossiping requirements. Formally it will look as follows: $\tau(n, k) \leq \tau(n - 1, k) + 2(k + 1)$.

$$\tau(n, k) \leq \frac{n-1}{2} \lceil \log_2 (n - 1) \rceil + \frac{(n-1)k}{2} + 2(k + 1), \tag{2.2}$$

where n is odd.

3. CONSTRUCTION OF FAULT-TOLERANT GOSSIP GRAPHS BASED ON COMBINED GRAPHS

In this section we are going to present a new method of constructing fault-tolerant gossip graphs which is based on the unification of several graphs. This method of constructing is preferable when $k << n$ and holds for all graphs with $n > 5$ vertices. Let's denote $G = (V, E)$ as a k fault-tolerant gossip graph with the vertex set V and the edge set E. We will present it as a combination of several graphs $G = H_1 + H_2 + H_3$. Let's denote as $\rho(G)$ linear ordering of edges in G.

At first let's consider cases when n is odd and k is even. In constructing of H_1 we have to make successive calls including all vertices. If we number all vertices from 1 to n , then we have to make new edges from i to $i{+}1$ vertex where $i = 1, 2, \dots, n - 1$. After making these edges we have to make one more edge which will connect the 1-st and $(n - 1)$-th vertices.

After constructing H_1 we have to construct a new graph(H_2). The construction of H_2 should be in the following way: its edges should connect the n-th ver- tex to all the other vertices in the following sequence: ($n{-}3$, $n{-}5, n{-}7, \dots, n{-}1, n{-}4, n{-}6, n{-}8, \dots, n{-}2$).

The last step of constructing G is the construction of H_3. In this step we have to make new successive edges that will connect the i-th and $(i - 1)$-th vertices $(i = n - 1, n - 2, \dots, 2)$.

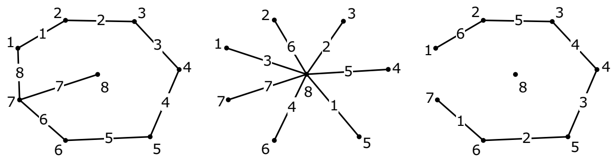

Figure 1. H_1,H_2 and H_3 graphs

After these 3 steps we will obtain $k = 2$ fault-tolerant gossip graph.

Suppose we have to construct a k fault-tolerant gossip graph with n vertices where k is even and n is odd. To achieve our target at first we should construct a 2 fault-tolerant gossip graph and then increase the level of fault-tolerance to k. For this we need to add new H_2 and H_3 graphs to G, but with one difference - the edges of new H_2 must connect the n-th vertex to other vertices in the following sequence: $(n - 5, n - 7, \dots, n - 1, n - 4, n - 6, n - 8, \dots, n - 2)$. So, repeating this action every time we increase the level of fault-tolerance with two until it reaches to k.

In case when n is even we have to change only the edge sequence in H_2 into the following: $(n{-}3, n{-}5, n{-}7, \dots, n{-}2, n{-}4, n{-}6, n{-}8, \dots, n{-}1)$.

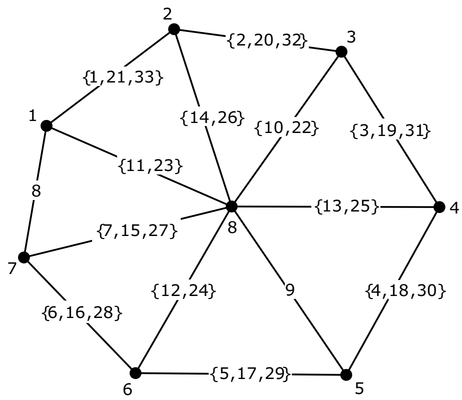

Figure 2. k=4 fault-tolerant gossip graph

Let's count how many edges we made when k was even. In H_1 we made $(n - 1) + 1$ edges, in H_2 graphs we made all together $k(n - 2)/2 + 1$ edges and in H_3 graphs we made $k(n - 2)/2$ edges. So, for general cases when k is even we obtain the following result: $\tau(n, k) \leq (n - 2) + 2 + k(n - 2)/2 + 1 + k(n - 2)/2 = (k + 1)(n - 2) + 3$.

Now consider cases when k is odd ($k \geq 3$). In such cases we recommend to construct a 3 fault-tolerant gossip graph and then increase the level of fault-tolerance until achieving k. The deviation from cases when k is even is the following: after obtaining the graph $H_1 + H_2$ we have to add a new H_2 graph, whose edges must connect the n-th vertex with all others in the following sequence: $(n-3, n-5, n-7, \dots, n-1, n-4, n-6, n-8, \dots, n-2)$ in case n is odd and in the following: $(n-3, n-5, n-7, \dots, n-2, n-4, n-6, n-8, \dots, n-1)$ for even n. After this step the H_3 graph must be added and all together they will form a 3 fault-tolerant gossip graph. So, repeating adding new H_2 and H_3 graphs every time we increase the level of fault-tolerance with two until it reaches k. Note, that in repeating action the H_2 graph must change it's edge sequence into $(n-5, n-7, \dots, n-1, n-4, n-6, n-8, \dots, n-2)$ if n is odd and into $(n-5, n-7, \dots, n-2, n-4, n-6, n-8, \dots, n-1)$ in case if n is even.

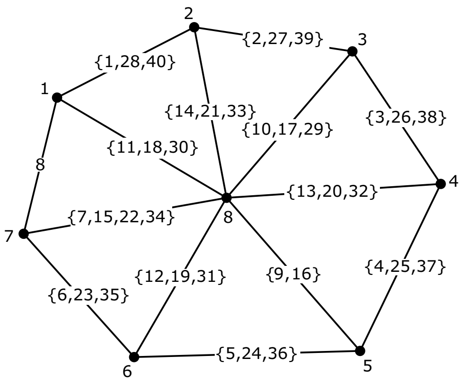

Figure 3. k=5 fault-tolerant gossip graph

Let's count the number of edges of G in case when k is odd. In H_1 there are $(n-1)+1$ edges and the sum of edges of all H_2 and H_3 graphs is $k(n-2)+2$. So, for odd k our result is $(n-2)+2+(n-2)k+2 = (k+1)(n-2)+4$.

Consider any two distinct vertices v_i and v_j in communication network G. It is an easy exercise for the reader to verify that there exist $k+1$ pairwise edge disjoint ascending paths from v_i to v_j.

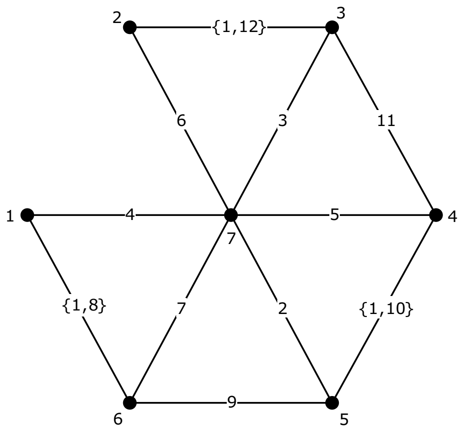

Figure 4. k=1 fault-tolerant gossip graph

At the end let's consider cases when $k = 1$ and $k = 2$. In these cases the construction of a graph becomes more simple. We will present it here without considering details. Just note, that $\tau(n,1) \leq 2n - 3 + \lfloor \frac{n}{2} \rfloor$ and $\tau(n,2) \leq 2n - 3 + n$. So, as we can see, it confirms the assumption of Hasunuma and Nagamochi $\tau(n,k) \leq \frac{nk}{2} + O(n)$.

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