Turning Points in Hamiltonian Graph Theory – A Survey

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ABSTRACT

The development of hamiltonian graph theory has undergone a natural growth and evolution. In the last 30-35 years, radical turnings occur in this area: from simplicity to complexity, from gradual small changes (more than 98-99% of all developments) to much bigger changes, from particular values to parameters. In this paper we try to reveal these turnings and specify their basis by focusing only on sharp results. For this purpose, we classify the basic research objects (Hamilton cycle, longest cycle, dominating cycle, and so on) of the area by their complexity levels, as well as the basic research tools (the number of edges, minimum degree, connectivity, toughness, and so on) having certain impact on research objects. The turnings in hamiltonian graph theory are related to generalized Hamilton cycles (including Hamilton and dominating cycles as special cases), vertex connectivity invariant, binding number, independence number and structures outside of cycles. The results related to generalized Hamilton cycles and structures outside of cycles are exceptionally due to the author. The most important results related to connectivity are also due to the author.

Keywords

Hamilton cycle, hamiltonian graph theory, turning points.

1. INTRODUCTION

Among various research objects in hamiltonian graph theory, Hamilton cycle is a starting point introduced by Kirkman (1855) and Hamilton (1856), independently. A Hamilton cycle of a graph is a cycle which passes through every vertex of the graph exactly once, and a graph is hamiltonian if it contains a Hamilton cycle. Classic hamiltonian problem; determining when a graph contains a Hamilton cycle, is one of the most central notions in graph theory and is one of the most attractive and most investigated problems among NP-complete problems.

The study of Hamilton cycles has been fueled also by practical applications and by the issue of complexity. In complexity theory, P denotes the set of polynomially solvable problems and NP denotes the set of nondeterministic polynomially solvable problems.

P versus NP problem: Does **P=NP**? (millennium prize problem).

A longest cycle is the next research object in hamiltonian graph theory, extending the notion of Hamilton cycle for a special case when graph is nonhamiltonian. Longest cycle problem; determining when a graph contains a cycle of length at least a given number, contains the classic hamiltonian problem as a special case. The main product of investigations concerning longest cycle problem consists of various lower bounds for the circumference (the length of a longest cycle).

Dominating cycle is another extension of Hamilton cycle, becoming the third important research object in the area. By the definition, a cycle C of a graph G is said to be a dominating cycle if G\C is edgeless.

Furthermore, PD_λ and CD_λ-cycles are introduced for the purpose to include each cycle of a graph for appropriate λ, since Hamilton and dominating cycles do not exhaust all possible kinds of cycles in a graph. For a given integer $\lambda \in \{0,1,...\}$, a cycle C in a graph G is called PD_λ-cycle if every path in G of length at least λ has a vertex in common with C. The vertices and edges in a graph can be considered as cycles of lengths 1 and 2, respectively. For a given integer $\lambda \in \{1,2,...\}$, a cycle C in G is called CD_λ-cycle if every cycle in G of length at least λ has a vertex in common with C. In particular, PD_0 and CD_1-cycles are Hamilton cycles. Furthermore, PD_1 and CD_2-cycles are dominating cycles. PD_λ and CD_λ-cycles can be considered as missing elements in the list of the main research objects. By the definition, the structures of PD_λ and CD_λ-cycles gradually become more complicated.

In sum, we have the following basic research objects in Hamiltonian graph theory.

The basic research objects: longest cycle; PD_λ and CD_λ-cycles, including Hamilton and dominating cycles as special cases.

Further developments in the area need the following additional elements.

Additional research objects: shortest cycle; arbitrary cycles; 2-factors; r-ordered Hamilton cycles; cycles containing specified elements: vertices, independent edges, disjoint paths, combination of paths and vertices (linear forests); and so on.

Noncyclic research objects: Hamilton path; longest path; spanning threes with minimum number of leafs.

Now we turn to research tools – graph characteristics having certain impact on research objects.

The basic research tools: the number of edges; minimum degree; connectivity; structures outside of cycles; binding number; independence number; toughness; forbidden subgraphs.

Advanced research tools: degree sequences, degree sums, neighborhood unions, generalized degrees, and so on.

The main goal of hamiltonian graph theory: study of research objects by means of research tools, involving them into certain relations in forms of particular values (first stage) or parameters (last stage).

Simple research tools: the number of edges, minimum degree and its various extensions of local nature; they can be tested by simple algorithms.
Complicated research tools: connectivity invariant and the binding number; they can be tested in polynomial time.
Hard research tools: independence number, toughness, structures outside of cycles and forbidden subgraphs; there are not known polynomial time algorithms for their testing.
Dynamics of developments in the area. The development of hamiltonian graph theory has undergone a natural growth and evolution. In the last 30-35 years, radical turnings occur in this area.
Slow developments: gradual small changes - more than 98-99% of all developments. This is a stage for experience accumulation and proof technique improvements. Slow developments are related to simple research tools (the number of edges, minimum degree and its various extensions), as well as complicated tools for special cases (2-connectivity, 3-connectivity, 1-toughness, H-free graphs, R,H-free graphs, and so on).
Turnings: sharp changes in relatively short periods of time from simplicity to complexity, from superficial observations to global observations, from gradual small changes to much bigger changes, from particular values to parameters.

The first group of turnings is related to research objects of the area (generalized Hamilton cycles, including Hamilton and dominating cycles as special cases) whose structures gradually become more complicated and further developments need radical revision of research approaches. The second group of turnings is related to research tools having different impact on research objects according to their three qualitative different complexity levels.

In this paper we try to reveal radical turnings in the area and specify their basis by focusing only on sharp results. These turnings are based on generalized Hamilton cycles, vertex connectivity invariant, binding number, independence number and structures outside of cycles. The results related to generalized Hamilton cycles and structures outside of cycles exceptionally are due to the author. The most important results related to connectivity are also due to the author. For the binding number a single result is known (Woodall [28]).

2. NOTATIONS AND DEFINITIONS

We consider only finite undirected graph without loops and multiple edges. A good reference for undefined terms is [2].

Let G be a graph with vertex set V(G) and edge set E(G). Throughout the paper, each vertex and edge of G can be interpreted as cycles of lengths 1 and 2, respectively. For a given longest cycle C in G, we denote by $\overline{p}$ and $\overline{c}$ the lengths of a longest path and a longest cycle in G\C, respectively.

We use n,δ,α,c to denote the order (the number of vertices), minimum degree, independence number (maximum number of mutually nonadjacent vertices) and the circumference (the length of a longest cycle) of G. The connectivity κ is defined to be the minimum number of vertices whose removal disconnects G or reduces it to a single vertex K_1. A graph is called s-connected if $s \le \kappa$. Let s(G) denotes the number of components of G. A graph G is t-tough if $|S| \ge ts(G\setminus S)$ for every subset $S \subset V(G)$ with $s(G\setminus S) > 1$ The toughness of G, denoted by τ, is the maximum value of t for which G is t-tough (taking $\tau(K_n)=\infty$ for all $n \ge 1$). The binding number $b(G)$ is defined as follows:

$$b(G)=\min_X\left\{\frac{|N(X)|}{|X|}:\varnothing \ne X \subseteq V(G)\right\},$$

where $N(X)=\bigcup_{x\in X} N(x)$. Put

$$\sigma_t=\min\left\{\sum_{i=1}^t d(v_i) \mid \{v_1,v_2,...,v_t\} \textit{ is independent}\right\}.$$

3. ON BASIC RESEARCH TOOLS

Connectivity. By many graph theorists, connectivity is at the heart of all path and cycle questions. 2-connectivity is a necessary condition for a graph to be hamiltonian and occurs in majority of hamiltonian results as one of the main conditions. Connectivity can be tested in polynomial time [8].
Structures outside of cycles. Since 1952, the structures outside of cycles are the main tools using to construct long cycles and estimate their lengths. In final results, they are mainly invisible. The first result involving structures outside of cycles in forms of parameters appeared in 1998.

The first idea, that arises when we try to construct long cycles in a graph G, is the following: choose an initial cycle C_1 in G and try to replace it by a longer cycle by using $G\setminus C_1$-structures (long paths, long cycles and so on). If the degrees of vertices and the connectivity are large enough then we have a good chance to find cycles longer than C_1 by replacing some segments of C_1 with some paths passing through $G\setminus C_1$. If we have found such cycle C_2 then we can repeat the same algorithm for C_2. Actually, this iterative algorithm is fallen in the base of all hamiltonian results.
Binding number. By the definition, the binding number $b(G)$ is a ratio of two different graph characteristics. The condition $b(G) \ge 3/2$ is similar to relations between two different parameters such as $\delta \ge n/2$ and $\kappa \ge \alpha$, having much more impact on cycle structures. The binding number can be tested in polynomial time [6].
Toughness. The most challenging conjecture in hamiltonian graph theory is still open: is there a finite constant t_0 such that every t_0-tough graph is hamiltonian (Chvátal [5]). The problem of determining of the complexity of recognizing t–tough graphs for any positive rational number t is NP-hard [1]. There are no sharp results involving τ as a parameter. A number of sharp results are known for 1-tough and $(1+\varepsilon)$-tough graphs, slightly improving some well-known results by replacing the connectivity conditions with 1-tough or $(1+\varepsilon)$-tough conditions.
Independence number. Independence number α occurs mainly in a number of relations of the type $\delta \ge \alpha$ and $\kappa \ge \alpha$, where the main performers are δ and κ, repulsing the role of α to the second plan.
Forbidden subgraphs. Forbidden subgraphs can essentially change the structure of a graph. For example, P_2-free graphs (graphs having no P_2 as an induced subgraph) are empty graphs, while the connected components of P_3-free graphs are complete graphs.

It is known that if 2-connected graph G is R-free then G is hamiltonian if and only if $R = P_3$. The second nontrivial sharp result states that if 2-connected graph G is R,S-free then G is hamiltonian if and only if $R = K_{1,3}$ and S belongs to a well defined class of graphs. A similar result is developed for forbidden triples with much more complicated description of graphs. So, $H_1, H_2, ..., H_m$-free graphs are completely studied only for m=1,2,3.

The second approach is based on a combination of forbidden subgraphs with other types of conditions involving minimum degree, connectivity, toughness and so on. Similar results are established for $K_{1,3}$-free graphs, but they are far from to be best possible.

4. CARDINAL TURNINGS

The first result involving connectivity κ and independence number α as parameters, was established in 1972.

Theorem 1 (Chvátal and Erdös, 1972) [4]
Every graph is hamiltonian if $\alpha \le \kappa$.

The proof of Theorem 1 is quite simple and is used in many other proofs as standard arguments.

The long cycle's version of Theorem 1 was established in 1994.

Theorem 2 (Kouider, 1994) [13]
In every graph, $c \ge n / \lceil \alpha / \kappa \rceil$.

The next theorem includes connectivity as a parameter.

Theorem 3 (Bondy, 1980) [3]
Every s-connected graph is Hamiltonian if

$$\frac{1}{s+1}\sigma_{s+1} \ge \frac{n}{2}.$$

The following two theorems include independence number as a parameter.

Theorem 4 (Nash-Williams, 1971) [15]
Every 2-connected graph with $\delta \ge \max\{(n+2)/3, \alpha\}$ is hamiltonian.

Theorem 5 (Jung, 1978) [11]
Every 3-connected graph with $\delta \ge \alpha$ either is hamiltonian or $c \ge 3\delta - 3$.

In 1981, connectivity appeared with minimum degree δ.

Theorem 6 (Nikoghosyan, 1981) [10], [17], [18]
Every 2-connected graph with $\delta \ge (n+\kappa)/3$ is hamiltonian.

Theorem 7 (Nikoghosyan, 1981) [17], [18]
Every 3-connected graph either is hamiltonian or $c \ge 3\delta - \kappa$.

Similar theorems were developed for dominating cycles.

Theorem 8 (Lu, Liu and Tian, 2005) [14]
All longest cycles in 3-connected graphs are dominating if $\delta \ge (n+2\kappa)/4$.

Theorem 9 (Nikoghosyan, 2009) [26]
Every 4-connected graph either has a dominating cycle or $c \ge 4\delta - 2\kappa$.

Using the additional condition $\delta \ge \alpha$, the minimum degree condition in Theorem 6 can be essentially relaxed and the lower bound in Theorem 7 can be essentially improved.

Theorem 10 (Nikoghosyan, 1984) [19], [21]
Every 3-connected graph is hamiltonian if $\delta \ge \max\{(n+2\kappa)/4, \alpha\}$.

Theorem 11 (Nikoghosyan, 1985) [20]
Every 4-connected graph with $\delta \ge \alpha$ either is hamiltonian or $c \ge 4\delta - 2\kappa$.

Theorem 8 is sharp only for $\kappa = 3$. Theorem 9 is sharp only for $\kappa = 4$. The final versions of Theorems 8 and 9 are sharp for each value of κ.

Theorem 12 (Yamashita, 2008) [29]
All longest cycles in a 3-connected graph are dominating if $\delta \ge (n+\kappa+3)/4$.

Theorem 13 (Nikoghosyan and Nikoghosyan, 2011) [16]
In every 4-connected graph, either all longest cycles are dominating or $c \ge 4\delta - \kappa - 4$.

Theorem 8 is sharp for $\kappa = 3$, and Theorem 9 is sharp for $\kappa = 4$. The final versions of Theorems 10 and 11 are sharp for each κ.

Theorem 14 (Yamashita, 2008) [29]
Every 3-connected graph is hamiltonian if $\delta \ge \max\{(n+\kappa+3)/4, \alpha\}$..

Theorem 15 (Nikoghosyan and Nikoghosyan, 2011) [16]
Every 4-connected graph with $\delta \ge \alpha$ either is hamiltonian or $c \ge 4\delta - \kappa - 4$.

In the next theorem, connectivity appears in forms of parameter λ. When $\lambda = 1$, we have a well-known theorem by Nash-Williams [15].

Theorem 16 (Fraisse, 1986) [9]
Let G be a $(\lambda+1)$-connected graph, where λ is a positive integer. Then G is hamiltonian if

$$\delta \ge \max\left\{\frac{n+2}{\lambda+2} + \lambda - 1, \alpha + \lambda - 1\right\}.$$

The long cycles version of Theorem 16 contains Jung's [11] theorem as a special case when $\lambda = 1$.

Theorem 17 (Nikoghosyan, 2009) [25]
If G is a $(\lambda+2)$-connected graph with $\delta \ge \alpha + \lambda - 1$ for some positive integer λ then G either is hamiltonian or $c \ge (\lambda+2)(\delta-\lambda)$.

The next two theorems contain Dirac's two well-known theorems [7] concerning Hamilton cycles when $\lambda = 1$, and two theorems by Nash-Williams [15] and Jung [12] concerning dominating cycles when $\lambda = 2$.

Theorem 18 (Nikoghosyan, 2009) [25]
Let G be a λ-connected graph for some positive integer λ. Then all longest cycles in G are $CD_{\min\{\lambda,\delta-\lambda+1\}}$-cycles if $\delta \geq (n+2)/(\lambda+1)+\lambda-2$.

Theorem 19 (Nikoghosyan, 2009) [25]
If G is a $(\lambda+1)$-connected graph for some positive integer λ then either all longest cycles in G are $CD_{\min\{\lambda,\delta-\lambda\}}$-cycles or $c \geq (\lambda+1)(\delta-\lambda+1)$.

The next three theorems are the first results involving $\overline{p}$ and $\overline{c}$ as parameters.

Theorem 20 (Nikoghosyan, 1998) [22], [27]
If C is a longest cycle in a graph then $|C| \geq (\overline{p}+2)(\delta-\overline{p})$.

Theorem 21 (Nikoghosyan, 2000) [23], [27]
If C is a longest cycle in a graph then $|C| \geq (\overline{c}+1)(\delta-\overline{c}+1)$:

Theorem 22 (Nikoghosyan, 2000) [24]
Let G be a 2-connected graph and let C be a longest cycle in G. If $\overline{c} \geq \kappa$ then

$$|C| \geq \frac{(\overline{c}+1)\kappa}{\overline{c}+\kappa+1}(\delta+2).$$

The last theorem includes $b(G)$ as a parameter.

Theorem 23 (Woodall, 1973) [28]
Every graph G with $b(G) \geq 3/2$ is hamiltonian.

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