On the achromatic indices of graph products

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ABSTRACT

A proper edge-coloring of a graph G with colors $1,\dots,t$ is called a complete edge $t-$coloring if for every pair of colors i and j, there are two edges with a common vertex, one colored by i and the other colored by j. The largest value of t for which G has a complete edge $t-$coloring is called the achromatic index $\psi'(G)$ of G. In this paper achromatic indices of various graph products are investigated.

Keywords

Achromatic number, achromatic index, complete coloring, graph products.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected, connected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The maximum degree of a vertex in G is denoted by $\Delta(G)$, the chromatic number of G by $\chi(G)$, and the chromatic index of G by $\chi'(G)$. We use the standard notation K_n for the complete graph on n vertices. For a graph G, by $\overline{G}$ and $L(G)$ we denote the complement and the line graph of the graph G, respectively.

Let G and H be two graphs.

The Cartesian product $G\Box H$ is defined as follows:

$$V(G\Box H)=V(G)\times V(H),$$

$$E(G\Box H)=\{(u_1,v_1)(u_2,v_2)\mid u_1=u_2 \text{ and } v_1v_2\in E(H) \text{ or } v_1=v_2 \text{ and } u_1u_2\in E(G)\}.$$

The tensor (direct) product $G\times H$ is defined as follows:

$$V(G\times H)=V(G)\times V(H),$$

$$E(G\times H)=\{(u_1,v_1)(u_2,v_2)\mid u_1u_2\in E(G) \text{ and } v_1v_2\in E(H)\}.$$

The strong tensor (semistrong) product $G\otimes H$ is defined as follows:

$$V(G\otimes H)=V(G)\times V(H),$$

$$E(G\otimes H)=\{(u_1,v_1)(u_2,v_2)\mid u_1u_2\in E(G) \text{ and } v_1v_2\in E(H) \text{ or } v_1=v_2 \text{ and } u_1u_2\in E(G)\}.$$

The strong product $G\boxtimes H$ is defined as follows:

$$V(G\boxtimes H)=V(G)\times V(H),$$

$$E(G\boxtimes H)=\{(u_1,v_1)(u_2,v_2)\mid u_1u_2\in E(G) \text{ and } v_1v_2\in E(H) \text{ or } u_1=u_2 \text{ and } v_1v_2\in E(H) \text{ or } v_1=v_2 \text{ and } u_1u_2\in E(G)\}.$$

The lexicographic product (composition) $G[H]$ is defined as follows:

$$V(G[H])=V(G)\times V(H),$$

$$E(G[H])=\{(u_1,v_1)(u_2,v_2)\mid u_1u_2\in E(G) \text{ or } u_1=u_2 \text{ and } v_1v_2\in E(H)\}.$$

The terms and concepts that we do not define can be found in [6,15].

A proper vertex $t-$coloring of a graph G is a mapping $\alpha : V(G)\rightarrow\{1,\dots,t\}$ such that for any $uv\in E(G)$, $\alpha(u)\neq\alpha(v)$. The chromatic number $\chi(G)$ is the smallest value of t for which G has a proper vertex $t-$coloring. A proper vertex $t-$coloring of a graph G is a complete vertex $t-$coloring of a graph G if for every pair of colors i and j, there is an edge $uv\in E(G)$ such that $\alpha(u)=i$ and $\alpha(v)=j$. The achromatic number $\psi(G)$ of G is the largest value of t for which G has a complete vertex $t-$coloring. The achromatic number of graphs was introduced by Harary and Hedetniemi in [7]. In [8], Harary, Hedetniemi and Prins showed that for any graph G if $\chi(G)\leq t\leq\psi(G)$, then G has a complete vertex $t-$coloring. In general, it is known that the problem of the determining of the achromatic number is $NP-$complete for bipartite graphs, cographs, interval graphs, and even for trees [1,5,14]. The achromatic numbers of graph operations were considered by Hell and Miller in [9], where they proved that for graphs G and H, $\psi(G\times H)\geq\psi(G)+\psi(H)$ except for some special cases. The achromatic numbers of the Cartesian products of graphs were considered by Chiang and Fu in [3], where the authors proved the following.

Theorem 1. If $\psi(G)=m$ and $\psi(H)=n$, then

$$\psi(G\Box H)\geq\psi(K_m\Box K_n)\geq\begin{cases} m+n-1, & \text{if } n>m=2 \text{ or } m=n>2,\\ 2n-\left\lceil\dfrac{n}{m-1}\right\rceil, & \text{if } n>m>2.\end{cases}$$

In the same paper, it was proved that $\psi(K_2\Box K_n)=n+1$ if $n\geq 3$, and $\psi(K_3\Box K_3)=5$, $\psi(K_3\Box K_n)=\left\lfloor\frac{3n}{2}\right\rfloor$ if $n\geq 4$. In [10-12], the achromatic numbers $\psi\left(K_4 \square K_n\right)$ and $\psi\left(K_5 \square K_n\right)$ were determined. In general, the achromatic number of the Cartesian product of K_m and K_n is unknown.

A proper edge-coloring of a graph G with colors $1,\dots,t$ is called a complete edge $t-$coloring if for every pair of colors i and j, there are two edges with a common vertex, one colored by i and the other colored by j. The achromatic index $\psi'(G)$ of G is the largest value of t for which G has a complete edge $t-$coloring. Clearly, for any graph G, $\psi'(G)=\psi\left(L(G)\right)$. In [2,4,13], the achromatic indices of complete and complete multipartite graphs are investigated. In this paper we investigate the achromatic indices of various graph products.

2. MAIN RESULTS

First we consider achromatic numbers and indices of Cartesian products of graphs, and we prove the following results.

Theorem 2. If $\psi\left(G\right)=\psi\left(H\right)=n$, then

$$\psi\left(G \square H\right) \geq \psi'\left(K_n\right)+1.$$

Theorem 3. For any graphs G and H,

$$\psi'\left(G \square H\right) \geq \psi'\left(G\right)+\psi'\left(H\right).$$

Next we consider achromatic indices of the tensor products of graphs, and we improve the result of Hell and Miller for line graphs.

Theorem 4. For any graphs G and H,

$$\psi'\left(G\times H\right) \geq \psi'\left(G\right)\cdot\psi'\left(H\right).$$

We also obtain a similar result for strong tensor products of graphs.

Theorem 5. For any graphs G and H,

$$\psi'\left(G\otimes H\right) \geq \psi'\left(G\right)+\psi'\left(G\right)\cdot\psi'\left(H\right).$$

Next we investigate achromatic indices of the strong products of graphs. In particular, we prove the following result.

Theorem 6. For any graphs G and H,

$$\psi'\left(G \boxtimes H\right) \geq \psi'\left(G\right)+\psi'\left(H\right)+\psi'\left(G\right)\cdot\psi'\left(H\right).$$

Corollary 1. For any $m,n\in\mathbb{N}$,

$$\psi'\left(K_{m\cdot n}\right) \geq \psi'\left(K_m\right)+\psi'\left(K_n\right)+\psi'\left(K_m\right)\cdot\psi'\left(K_n\right).$$

Corollary 2. For any $n\in\mathbb{N}$,

$$\psi'\left(K_{2\cdot n}\right) \geq 2\cdot\psi'\left(K_n\right)+1.$$

Finally we investigate achromatic indices of the composition of graphs. In particular, we prove the following results.

Theorem 7. For any graph G and $n\in\mathbb{N}$,

$$\psi'\left(G\left[\overline{K_n}\right]\right) \geq \psi'\left(G\right)+\psi'\left(G\right)\cdot\psi'\left(K_n\right).$$

Theorem 8. If H is a regular graph, $\chi'(H)=\Delta(H)$ and $\left|V(H)\right|=n$, then for any graph G,

$$\psi'\left(G\left[H\right]\right) \geq \psi'\left(G\right)+\chi'\left(H\right)+\psi'\left(G\right)\cdot\psi'\left(K_n\right).$$

Corollary 3. If H is a regular bipartite graph and $\left|V(H)\right|=n$, then for any graph G,

$$\psi'\left(G\left[H\right]\right) \geq \psi'\left(G\right)+\chi'\left(H\right)+\psi'\left(G\right)\cdot\psi'\left(K_n\right).$$

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