

Interval cyclic edge-colorings of graphs

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ABSTRACT

A proper edge-coloring of a graph G with colors $1, \dots, t$ is called an interval cyclic t -coloring if all colors are used, and the edges incident to each vertex $v \in V(G)$ are colored with $d_G(v)$ consecutive colors by modulo t , where $d_G(v)$ is the degree of the vertex v in G . In this paper some properties of interval cyclic edge-colorings are investigated. Also, we give some bounds for the greatest possible number of colors in such colorings for complete, complete bipartite graphs and hypercubes.

Keywords

Edge-coloring, interval edge-coloring, connected graph, bipartite graph, regular graph.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected, connected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G , respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of a vertex in G by $\Delta(G)$, the chromatic index of G by $\chi'(G)$, and the diameter of G by $\text{diam}(G)$. We use the standard notations C_n , K_n and Q_n for the simple cycle, complete graph on n vertices and the hypercube, respectively. We also use the standard notations $K_{m,n}$ and $K_{m,n,l}$ for the complete bipartite graph and tripartite graph, one part of which has m vertices, the other part has n vertices and the third part has l vertices. The terms and concepts that we do not define can be found in [3,17].

An interval t -coloring [1] of a graph G is a proper edge-coloring of G such that all colors are used, and the edges incident to each vertex $v \in V(G)$ are colored by $d_G(v)$ consecutive colors. A graph G is interval colorable if it has an interval t -coloring for some positive integer t . The set of all interval colorable graphs is denoted by \mathfrak{I} . For a graph $G \in \mathfrak{I}$, the least and the greatest values of t for which G has an interval t -coloring are denoted by $w(G)$ and $W(G)$, respectively.

The concept of interval edge-coloring of graphs was introduced by Asratian and Kamalian [1,2]. In [1,2], they proved that if G is interval colorable, then $\chi'(G) = \Delta(G)$. They also showed that if G is a triangle-free graph and $G \in \mathfrak{I}$, then $W(G) \leq |V(G)| - 1$. In [6,7], Kamalian investigated interval edge-colorings of complete bipartite graphs and trees. In particular, he proved that the complete bipartite graph $K_{m,n}$ has an interval t -coloring if and only if $m + n - \text{gcd}(m, n) \leq t \leq m + n - 1$, where $\text{gcd}(m, n)$ is the greatest common divisor of m and n . Later, Kamalian [7] obtained an upper bound on $W(G)$ for an interval colorable graph G depending on the number of vertices of G . In particular, he proved that if G is a connected graph and $G \in \mathfrak{I}$, then $W(G) \leq 2|V(G)| - 3$. Clearly, this bound is sharp for the complete graph K_2 , but if $G \neq K_2$, then this upper bound can be improved to $W(G) \leq 2|V(G)| - 4$ [5]. For an r -regular graph G , Kamalian and Petrosyan [9] showed that if $G \in \mathfrak{I}$ and G with at least $2r + 2$ vertices, then $W(G) \leq 2|V(G)| - 5$. For a planar graph G , Axenovich [4] showed that if $G \in \mathfrak{I}$, then $W(G) \leq \frac{11}{6}|V(G)|$. In [13], Petrosyan investigated interval edge-colorings of complete graphs and hypercubes. In particular, he proved that if $n \leq t \leq \frac{n(n+1)}{2}$, then the hypercube Q_n has an interval t -coloring. Recently, Petrosyan, Khachatryan and Tananyan [14] showed that the hypercube Q_n has an interval t -coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$. In [15], Sevast'janov proved that it is an NP -complete problem to decide whether a bipartite graph has an interval coloring or not.

An interval cyclic t -coloring [16] of a graph G is a proper edge-coloring of G such that all colors are used, and the edges incident to each vertex $v \in V(G)$ are colored with $d_G(v)$ consecutive colors by modulo t . A graph G is interval cyclic colorable if it has an interval cyclic t -coloring for some positive integer t . The set of all interval cyclic colorable graphs is denoted by \mathfrak{I}_c . For a graph $G \in \mathfrak{I}_c$, the least and the greatest values of t for

which G has an interval cyclic t -coloring are denoted by $w_c(G)$ and $W_c(G)$, respectively. Clearly, if $G \in \mathfrak{R}$, then $G \in \mathfrak{R}_c$ and

$$\chi'(G) \leq w_c(G) \leq w(G) \leq W(G) \leq W_c(G) \leq |E(G)|.$$

The concept of interval cyclic edge-coloring of graphs was introduced by de Werra and Solot [16]. In [16], they proved that if G is an outerplanar bipartite graph, then $G \in \mathfrak{R}_c$ and G has an interval cyclic t -coloring for any $t \geq \Delta(G)$. This type of coloring under the name of “ π -coloring” was also considered by Kotzig in [11], where he proved that every cubic graph has a π -coloring with 5 colors. In [8], Kamalian investigated interval cyclic edge-colorings of trees, where he showed that for any tree T , $T \in \mathfrak{R}_c$ and $w_c(T) = w(T)$, $W_c(T) = W(T)$. Also, Kamalian [10] considered interval cyclic edge-colorings of simple cycles C_n , where he proved that for any $n \geq 3$, $C_n \in \mathfrak{R}_c$ and $w_c(C_n) = \chi'(C_n)$, $W_c(C_n) = n$. Moreover, he determined all possible values of t , $w_c(C_n) \leq t \leq W_c(C_n)$, for which C_n has an interval cyclic t -coloring. Interval cyclic edge-colorings of graphs also were considered by Nadolski in [12], where he showed that if $G \in \mathfrak{R}$, then $G \in \mathfrak{R}_c$ and $w_c(G) = \Delta(G)$. Moreover, he proved [12] that if G is a connected graph with $\Delta(G) = 3$, then $G \in \mathfrak{R}_c$ and $w_c(G) \leq 4$.

In this paper some properties of interval cyclic edge-colorings are investigated. Also, we give some bounds for the greatest possible number of colors in such colorings for complete, complete bipartite graphs and hypercubes.

2. SOME GENERAL RESULTS

First, we give some upper bounds on $W(G)$ for interval cyclic colorable connected graphs G .

Theorem 1. If G is a connected graph and $G \in \mathfrak{R}_c$, then

$$W_c(G) \leq 1 + 2 \cdot \max_{P \in \mathcal{P}} \sum_{v \in V(P)} (d_G(v) - 1),$$

where \mathcal{P} is the set of all shortest paths in G .

Corollary 1. If G is a connected graph and $G \in \mathfrak{R}_c$, then

$$W_c(G) \leq 1 + 2 \cdot (\text{diam}(G) + 1)(\Delta(G) - 1).$$

Note that corollary 1 was first obtained by Nadolski in [12].

Theorem 2. If G is a connected bipartite graph and $G \in \mathfrak{R}_c$, then

$$W_c(G) \leq 1 + 2 \cdot \text{diam}(G)(\Delta(G) - 1).$$

Theorem 3. If G is a regular graph, then

1. $G \in \mathfrak{R}_c$ and $w_c(G) = \chi'(G)$,
2. if $G \in \mathfrak{R}$ and $\Delta(G) \leq t \leq W(G)$, then G has an interval cyclic t -coloring.

Note that Theorem 3 implies that $\mathfrak{R} \subset \mathfrak{R}_c$.

Before we formulate our next result we need some definitions.

Let T be a tree and $V(T) = \{v_1, v_2, \dots, v_n\}$, $n \geq 2$. Let $P(v_i, v_j)$ be a simple path joining v_i and v_j , $VP(v_i, v_j)$ and $EP(v_i, v_j)$ denote the sets of vertices and edges of the path, respectively.

For a simple path $P(v_i, v_j)$, define $LP(v_i, v_j)$ as follows:

$$LP(v_i, v_j) = |EP(v_i, v_j)| + \left| \left\{ uv \mid uv \in E(T), u \in VP(v_i, v_j), v \notin VP(v_i, v_j) \right\} \right|.$$

Let $F(T)$ be a set of pendant vertices of T . Define: $M(T) = \max_{u, v \in F(T)} LP(u, v)$. Let us define the graph \hat{T} as follows:

$$V(\hat{T}) = V(T) \cup \{w\}, w \notin V(T),$$

$$E(\hat{T}) = E(T) \cup \{wv \mid v \in F(T)\}.$$

Clearly, \hat{T} is a connected graph with $\Delta(\hat{T}) = |F(T)|$. Moreover, if T is a tree in which the distance between any two pendant vertices is even, then \hat{T} is a connected bipartite graph.

Theorem 4. If T is a tree and $|F(T)| > 2(M(T) + 2)$, then $\hat{T} \notin \mathfrak{R}_c$.

3. INTERVAL CYCLIC EDGE-COLORINGS OF COMPLETE, COMPLETE BIPARTITE GRAPHS AND HYPERCUBES

In [9], Petrosyan considered interval edge-colorings of complete graphs and hypercubes. In particular, he proved the following

Theorem 5. If $n = p2^q$, where p is odd and q is nonnegative, then $W(K_{2n}) \geq 4n - 2 - p - q$.

By Theorem 3, we have that if $n \in \mathbb{N}$, then $K_{2n}, K_{2n+1} \in \mathfrak{R}_c$ and $w_c(K_{2n}) = 2n - 1$, $w_c(K_{2n+1}) = 2n + 1$. By Theorem 5, we have that if $n = p2^q$, where p is odd and q is nonnegative, then $W_c(K_{2n}) \geq 4n - 2 - p - q$. Now we give a lower bound for $W_c(K_{2n+1})$.

Theorem 6. If $n \in \mathbb{N}$, then $W_c(K_{2n+1}) \geq 3n$.

Next we consider complete bipartite graphs $K_{m,n}$.

Theorem 7. If $\min\{m, n\} = 1$, then $K_{m,n} \in \mathfrak{R}_c$ and $w_c(K_{m,n}) = W_c(K_{m,n}) = m + n - 1$. If $\min\{m, n\} \geq 2$ and $\max\{m, n\} \leq t \leq m + n$, then $K_{m,n}$ has an interval cyclic t -coloring.

Corollary 2. If $\min\{m, n\} = 1$, then $K_{m,n} \in \mathfrak{R}_c$ and $w_c(K_{m,n}) = W_c(K_{m,n}) = m + n - 1$. If $\min\{m, n\} \geq 2$, then $K_{m,n} \in \mathfrak{R}_c$ and $w_c(K_{m,n}) = \max\{m, n\}$, $W_c(K_{m,n}) \geq m + n$.

We also consider complete tripartite graphs $K_{1,m,n}$.

Theorem 8. If $m, n \in \mathbb{N}$, then $K_{1,m,n} \in \mathfrak{R}_c$ and $W_c(K_{1,m,n}) \geq m + n + 1$.

In [14], Petrosyan, Khachatryan and Tananyan showed that for hypercubes Q_n , $W(Q_n) = \frac{n(n+1)}{2}$. This implies that if $n \in \mathbb{N}$, then $W_c(Q_n) \geq \frac{n(n+1)}{2}$. We show that $W_c(Q_1) = 1$, $W_c(Q_2) = 4$, $W_c(Q_3) = 8$ and $12 \leq W_c(Q_4) \leq 14$. On the other hand, by Theorem 2, and taking into account that $\text{diam}(Q_n) = \Delta(Q_n) = n$, we obtain $W_c(Q_n) \leq 2n(n-1) + 1$, hence $W_c(Q_n) = O(n^2)$.

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