On an approximation algorithm for MINLA problem restricted to some classes of graphs

Hovhannes Sargsyan
Yerevan State University
Yerevan, Armenia
e-mail:hovhannes.sargsyan@improvis.am

ABSTRACT

It is known that MINLA of transitive oriented, bipartite graphs is NP-complete [1]. In this article we present an approximation algorithm for MINLA of transitive oriented graphs that gives an optimal linear arrangement for the class of so called bipartite Γ-oriented graphs, which is a subset of the transitive oriented graphs.

Keywords

MINLA, graph arrangement, transitive oriented graphs

1. Introduction

The problem of minimum linear arrangement (MINLA) of oriented graphs is defined as follows:

Problem: For a given oriented graph $G(V,E)$ construct such one to one function $f\colon \mathrm{V} \to \{1,\dots,|\mathrm{V}|\}$ that the following two conditions are satisfied:

$$(\mathrm{u},\mathrm{v}) \in E(G) \quad f(\mathrm{u}) < f(\mathrm{v}) \qquad (1)$$

$$L(G,f) \equiv \sum_{(\mathrm{u},\mathrm{v})\in \mathrm{E}} (f(\mathrm{v}) - f(\mathrm{u})\,) \to \min \qquad (2)$$

The condition (1) is also called an acceptability condition, and any function which satisfies the condition (1) is also called a labeling function for graph G. It is known that MINLA of oriented graphs is NP-complete [2] and it remains NP-complete for transitive oriented, bipartite graphs [1]. We are going to propose an approximation algorithm for MINLA of transitive oriented graphs which gives an optimal labeling for the subset of transitive oriented, bipartite graphs.

2. Preliminaries

Let $G(V,E)$ be an oriented acyclic graph without loops and multiple edges. For each $v \in V(G)$ let's define sets of ancestors and predecessors as follows:

$$_v^- = \{u \in V \ / \ (u,v) \in E(G)\} \text{ and } \ _v^+ = \{u \in V \ / \ (v,u) \in E(G)\}.$$

It is not hard to see, that in the expression (2), for each $v \in V(G)$, the label $f\,(v)$ appears with the coefficient +1 exactly $|\Gamma_v^-|$ times and with the coefficient -1 exactly $|\Gamma_v^+|$ times. So, each vertex $v \in V(G)$ with the label $f\,(v)$ induces a length equal to $|\Gamma_v^-|\,f(v) - |\Gamma_v^+|\,f(v)$, and the overall length for the function f is equal to the sum of the lengths induced by each vertex:

$$L(G,f) = \sum_{(u,v)\in E(G} (\,f\,(v) - f\,(u)\,) = \sum_{v\in V} f(v)\big(|\Gamma_v^-| - |\Gamma_v^+|\big) \quad (3)$$

Let's define another function $c : V(G) \to Z$ as follows:

$$c(v) = |\Gamma_v^-| - |\Gamma_v^+| \text{ for each } v \in V(G)$$

Based on the previous definition, the right part of the equation (3) can be transformed as follows:

$$\sum_{(u,v)\in E(G)} (\,f\,(v) - f\,(u)\,) = \sum_{v\in V(G)} f\,(v)c(v)$$

Without loss of generality we can assume that the indexation of the vertices of the graph G satisfies the following condition:

$$c(v_1) \le c(v_2) \le \cdots \le c(v_{|V|})$$

It is known that for arbitrary sequences $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ and for any permutation $f\colon \{1,2,\dots n\} \to \{1,2,\dots,n\}$, the following inequality holds:

$$\sum_{i=1}^{n} a_{\mathrm{i}}\, b_i \ge \sum_{i=1}^{n} a_{f(i)} b_i \ge \sum_{i=1}^{n} a_{\mathrm{n+1-i}}\, b_i$$

In our case we have 2 numerical sequences F and C with the length $|V(G)|$ where $F_i = i$ and $C_i = c(v_i)$, $i = 1,2,\dots|V|$, and it is required to find such $f\colon \{1,2,\dots|V|\} \to \{1,2,\dots|V|\}$ that

$$\forall (v_i, v_j) \in E(G)\ f(i) < f(j)$$

and

$$\sum_{i=1}^{|V|} F_{f(i)}\, C_i \to \min \quad (4)$$

So the lower bound of the minimum value of the expression (4) can be reached on the permutation

$$f(i) = |V| + 1 - i\,, \qquad i = 1,2,\dots,|V| \ (5)$$

and equals to

$$\sum_{i=1}^{|V|}(n+1-i)\,c(v_i)$$

It is not difficult to see that the permutation (5) not always satisfies the condition (1) but the main idea that we can get from this is the following:

*Vertices with bigger values of the function **c** should get smaller labels.*

3. The Algorithm

Let us define a polynomial labeling algorithm for transitive oriented graphs which uses the idea mentioned above, and generates a labeling that satisfies the condition (1).

Labeling Algorithm

Input: A transitive oriented graph $G(V,E)$ with $V(G) = \{v_1, v_2, \dots, v_n\}$.

Output: $f\colon \{v_1, v_2, \dots, v_n\} \to \{1, 2, \dots, n\}$, which satisfies the acceptability condition.

Step 1. Calculate values of the function c for each v_i, sort them by the increasing order $c(v_{i_1}) \le c(v_{i_2}) \le \dots \le c(v_{i_n})$ and assign infinite labels to all vertices of G.

Step 2. Iteratively, for $j = 1, \dots, n$, assign the label j to the vertex v_{i_j}, $i.e. f(v_{i_j}) := j$, and start the label decreasing procedure on the substep 2.1

Substep 2.1 While there is a vertex u, such that $f\left(v_{i_j}\right) = f(u) + 1$ and $(u, v_{i_j}) \notin E(G)$, (**swapping condition**), $f(v_{i_j}) := f(v_{i_j}) - 1$ and $f(u) := f(u) + 1$.

Theorem 1.

For an arbitrary transitive oriented graph G, the output of *Labeling Algorithm* satisfies the condition (1).

Proof.

Let f be the function constructed by *Labeling Algorithm* applied to the graph $G(V,E)$. Note that relative positions of vertices $v_{i_1}, v_{i_2}, \dots, v_{i_j}$, labeled during j-th iteration of the step 2, will not change till the end of the algorithm , and let us prove the theorem by contradiction, supposing that $\exists\,(u', v') \in E(G)$ for which $f(u') > f(v')$. Let us define by $S(G,f)$ the set of edges $(u'', v'') \in E(G)$, for which the acceptability condition is violated and select in $S(G,f)$ such an edge (u,v) , for which

$$f(u) - f(v) = \min\{f(u') - f(v')/(u',v') \in S(G,f)\}.$$

There are 2 possible cases.

Case 1: The vertex u has been labeled by the step 2 before the vertex v. Suppose v has got its label at k-th iteration of the step 2. It means that at the beginning of the substep 2.1 $f(v) = k$ and $f(u) = r$, where $r < k$. Since relative positions of u and v will not change after k-th iteration , then $f(v)$ becomes smaller than $f(u)$ during k-th iteration , which means that a situation would arise where $f(v) = f(u) + 1$ and on the substep 2.1 the labels of these vertices should be swapped, which is impossible since u and v do not satisfy the swapping condition. Thus, we have obtained a contradiction in this case.

Case 2: The vertex u has been labeled by the step 2 after the vertex v. Suppose v has got its label at k-th iteration of the step 2. It is clear that during k -th iteration of the substep 2.1 and after it $f(u)$ remains greater than $f(v)$. It means that during the execution of the substep 2.1 a vertex $t \in V(G)$ was considered, such that $f(u) > f(t) > f(v)$ and $(t,u) \in E(G)$, otherwise the substep 2.1 would decrease the label of u and finally we would have $f(u) < f(v)$. Since we have that $(u,v) \in E(G)$ and G is a transitive oriented graph, then it follows from $(t,u) \in E(G)$ and $(u,v) \in E(G)$ that $(t,v) \in E(G)$. Since till the end of the algorithm the relative positions of the vertices u, t *and* v do not change we will have that $f(u) > f(t) > f(v)$ and since $(t,v) \in E(G)$, we will have that $(t,v) \in S(G,f)$. Recall that we have selected $(u,v) \in S(G,f)$ under the condition

$$f(u) - f(v) = \min\{f(u') - f(v')/(u',v') \in S(G,f)\}$$

but in this case we have $(t,v) \in S(G,f)$, and, consequently,

$$\min\{f(u') - f(v')/(u',v') \in S(G,f)\} > f(t) - f(v)$$

which is a contradiction, too.

So we have proved that the output of *Labeling Algorithm* satisfies the condition (1).

The theorem is proved.

Remark 1. It is easy to see that the complexity of the *Labeling Algorithm* is $O(|V|^2)$.

Remark 2. Note (Fig.1), that for general oriented graphs the *Labeling Algorithm* can give a labeling that does not satisfy the condition (1).

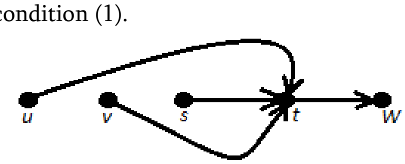

Fig. 1

3.1 Optimality of Labeling Algorithm for Γ-oriented Graphs

Let us define the subclass of Γ−oriented graphs, which is a subset of the set of transitive oriented graphs.

Definition.

An oriented acyclic graph $G(V, E)$ is called Γ−oriented if for any $u, v \in V(G)$ either $^{+}_{v} \subseteq {}^{+}_{u}$ or $^{+}_{u} \subseteq {}^{+}_{v}$.

It is easy to see that any Γ−oriented graph is transitive oriented and, therefore, our algorithm is applicable for these graphs. It is known [3] that for any Γ−oriented bipartite graph $G(V, E)$ with $V = X \cup Y$, $X = \{x_1, x_2, \dots, x_n\}$, $Y = \{y_1, y_2, \dots, y_m\}$, where $|\Gamma^+_{x_1}| \geq |\Gamma^+_{x_2}| \geq \cdots \geq |\Gamma_{x}{}^{+}{}_{n}|$, $|\Gamma_{y}{}^{-}{}_{1}| \leq |\Gamma_{y}{}^{-}{}_{2}| \leq \cdots \leq |\Gamma_{y}{}^{-}{}_{m}|$ (Fig 2) , there exists an optimal linear arrangement of the following kind:

$$x_n \ x_{n-1} \ \dots \ x_1 \ y_m \ y_{m-1} \ \dots \ y_1.$$

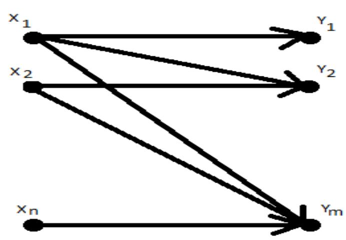

Fig 2. General representation of Γ−oriented bipartite graphs

Theorem 2.

Labeling Algorithm gives an optimal linear labeling for any Γ−oriented, bipartite graph $G(V, E)$.

Proof.

At first, the step 1 sorts vertices based on the definition of the function c as follows:

$$c(x_1) \leq \cdots \leq c(x_{n-1}) \leq c(x_n) < c(y_1) \leq c(y_2) \dots \leq c(y_m)$$

Taking into account that there is no edge between vertices of X and that the **swapping condition** always holds, we can conclude that for vertices from X we will have the following labels assigned to $x_1, x_2, \dots, x_n$.

$$f(x_i) = n + 1 - i, \qquad i = 1, \dots, n$$

Since $(x_1, y) \in E(G)$ for each $y \in Y$, then $n = f(x_1) < f(y)$. Since there is no edge between vertices of Y, then at the end of the algorithm we will have the following arrangement:

$$x_n \ x_{n-1} \dots x_1 \ y_m \ y_{m-1} \ \dots \ y_1$$

which is the optimal arrangement for G.

The theorem is proved.

So we have presented the idea to approximate MINLA of transitive oriented graphs and a polynomial algorithm based on that idea which gives an optimal solution for the subset of transitive oriented bipartite graphs. It is an open question whether MINLA is NP-complete for Γ−oriented graphs, and if there are other subclasses of transitive oriented graphs, for which MINLA can be solved in a polynomial time.

4. ACKNOWLEDGEMENT

The author thanks S. Ye. Markosyan and R. R. Kamalian for their attention to this work.

REFERENCES

1. H.E. Sargsyan, S.Ye. Markosyan. Minimum linear arrangement of the transitive oriented, bipartite Graphs, Proceedings of the Yerevan State University, Physical and Mathematical Sciences, 2012 № 2, p. 50–54

2. Even S., Shiloah Y. NP-Completeness of Several Arrangement Problems. Report 43.Dept. of Computer Science. Israel, Haifa: Technion, 1975

3. L. Rafayelyan, The Minimum Linear Arrangement Problem on Bipartite Γ–oriented graphs, Mathematical Problems of Computer Science 32, 23-26, 2009