Reconstruction of quantum information after the measurement

Sergey Poghosyan
Laboratory of Physics, Kochi University of Technology,
Tosa Yamada, Kochi 782-8502, Japan
e-mail: sergey.poghosyan@kochi-tech.ac.jp

Taksu Cheon
Laboratory of Physics, Kochi University of Technology,
Tosa Yamada, Kochi 782-8502, Japan
e-mail: taksu.cheon@kochi-tech.ac.jp

ABSTRACT

We consider weak value expansion of the Hermitian operator in terms of a set of operators formed from biorthogonal basis. The utility of the expansion is showcased with examples of spin one-half and spin one systems, where irreversible subset of stochastic matrices describing projective measurement on a mixed state is identified.

Keywords

Weak value, quantum information, bistochastic matrices.

1. INTRODUCTION

The concept of weak value, along with its experimental validation by weak measurements, has been around for almost quarter century [1, 2, 3, 4, 5]. It is based on the idea of physical value subjected to two successive measurements [6], which has been later extended to general multiple-time measurements [7]. The weak value formalism has a wide range of applications in various fields of quantum information theory. Particularly, it can be used to transfer quantum-communication protocols [8], describe entangled systems [9], reconstruct quantum optical states by weak measurements [10, 11] and clone quantum systems [12]. To many physicists' minds, however, the concept still entails an aura of mystery, and confusion over its physical interpretations [13, 14, 15] never seems to have been fully cleared.

In this article, we intend to reconcile this often mystified concept of weak value with the conventional orthogonal vector space formulation of quantum mechanics by introducing a *complete set* of weak values. We point out the existence of a set of operators defined by biorthogonal Hilbert vector bases, with which any Hermitian operator can be expanded, upon which the weak values emerge as the expansion coefficients.

Virtue of bringing in the whole set of weak values, is demonstrated by an example, in which a mixed quantum state undergoes a projective measurement that brings the system into another mixed state. This process is shown to be easily described by an unistochastic matrix, a "quantum" subset of stochastic matrices [16]. We identify the condition in which the unistochastic matrix is irreversible for the case of a spin one-half and spin one systems. We obtain the subset of all possible projective measurements, for which the history is erased, namely, the reconstruction of the original state is impossible after the measurement.

2. WEAK VALUE EXPANSION

Consider a Hermitian operator A on Hilbert space of dimension n. We attempt to represent A with two different orthonormal bases $\{|\psi_j\rangle, j = 1,...,n\}$ with $\langle\psi_i|\psi_j\rangle = \delta_{i,j}$ and $\{|\phi_\ell\rangle\, \ell = 1,...,n\}$ with $\langle\phi_k|\phi_\ell\rangle = \delta_{k,\ell}$. We assume that we have the property $\langle\phi_\ell|\psi_j\rangle \neq 0$ for all ℓ, j. Following Aharonov, Albert and Vaidman [1], let us define the weak value (A) of A by

$$(A)_{\ell,j} = \frac{\langle\phi_\ell| A |\psi_j\rangle}{\langle\phi_\ell|\psi_j\rangle}. \tag{1}$$

We will use a weak value expansion, which is, technically, just a simple modification of standard expansion of a Hermitian operator A with orthonormal states $\{|\psi_j\rangle\}$

$$A = \sum_{i,j} |\psi_i\rangle \langle\psi_i| A |\psi_j\rangle \langle\psi_j|, \tag{2}$$

using two sets of orthogonal states $\{|\psi_j\rangle, |\phi_\ell\rangle\}$ instead of one set. We also define the overlap matrices

$$\mu_{\ell,j} = |\langle\phi_\ell|\psi_j\rangle|^2. \tag{3}$$

Suppose Alice passes on a mixed state

$$\rho = \sum_j |\psi_j\rangle \rho_j^{(\psi)} \langle\psi_j| \tag{4}$$

to Bob, on which Bob performs a projective measurement using the basis $\{|\phi_\ell\rangle, \ell = 1,2,...,n\}$ and obtain the mixed state

$$\tau = \sum_\ell |\phi_\ell\rangle \tau_\ell^{(\phi)} \langle\phi_\ell|. \tag{5}$$

Let us ask how Bob can reconstruct the state ρ with the knowledge that Alice had obtained her state from a projective measurement in the basis $\{|\psi_j\rangle, j = 1,2,...,n\}$.

If the Alice's state is expressed in the basis $\{|\phi_\ell\rangle\}$, a generic representation with non-diagonal elements should be obtained, namely

$$\rho = \sum_{\ell,m} |\phi_\ell\rangle \rho_{\ell m}^{(\phi)} \langle\phi_m| \tag{6}$$

with $\rho_{\ell m}^{(\phi)} = \langle\phi_\ell| \rho |\phi_m\rangle$. After the projective measurement, only diagonal components of this expression remain, and we should have $\tau_\ell^{(\phi)} = \rho_{\ell\ell}^{(\phi)}$.

If we consider the *weak value of* ρ between states $|\phi_m\rangle$ and $|\psi_j\rangle$, then obtain

$$\frac{\langle\phi_m|\,\rho\,|\psi_j\rangle}{\langle\phi_m|\psi_j\rangle} = \rho_j^{(\psi)}, \tag{7}$$

for any m, since $\rho\,|\psi_j\rangle = \rho_j^{(\psi)}\,|\psi_j\rangle$, thus giving the formal answer to our question in terms of the weak values. In order to obtain the explicit expression of $\rho_j^{(\psi)}$ in terms of $\tau_m^{(\phi)}$, we rewrite this equation, by inserting the complete set $\sum_j |\psi_j\rangle\langle\psi_j|$ in front of ρ in the LHS, in the form

$$\langle\phi_m|\psi_j\rangle\,\rho_j^{(\psi)} - \sum_{\ell\neq m}\langle\phi_\ell|\psi_j\rangle\,\rho_{m\ell}^{(\phi)} = \langle\phi_m|\psi_j\rangle\,\tau_m^{(\phi)}, \tag{8}$$

which can be reformulated as N^2 linear equations indexed by (m,j);

$$A_{k,\ell}^{(m,j)} X_{k,\ell} = B^{(m,j)}, \tag{9}$$

for N^2 unknown variables

$$\begin{aligned} X_{k,\ell} &= \rho_k^{(\psi)} \qquad (k=\ell), \\ &= \rho_{k,\ell}^{(\phi)} \qquad (k\neq\ell) \end{aligned} \tag{10}$$

with

$$\begin{aligned} A_{k,\ell}^{(m,j)} &= \delta_{k,j}\delta_{\ell,j}\,\langle\phi_m|\psi_j\rangle - \delta_{m,k}(1-\delta_{\ell,m})\,\langle\phi_\ell|\psi_j\rangle\,, \\ B^{(m,j)} &= \langle\phi_m|\psi_j\rangle\,\tau_m^{(\phi)}. \end{aligned} \tag{11}$$

Clearly, this gives the solution to the problem of state reconstruction.

We can further multiply $\langle\psi_j|\phi_k\rangle$ to the above equation from the right and sum up by j to obtain

$$\sum_j \langle\phi_m|\psi_j\rangle\,\langle\psi_j|\phi_k\rangle\,\rho_j^{(\psi)} - (1-\delta_{mk})\rho_{mk}^{(\phi)} = \tau_m^{(\phi)}\delta_{mk}, \tag{12}$$

which splits into

$$\sum_j \mu_{mj}\rho_j^{(\psi)} = \tau_m^{(\phi)}, \tag{13}$$

$$\sum_j \langle\phi_m|\psi_j\rangle\,\langle\psi_j|\phi_k\rangle\,\rho_j^{(\psi)} = \rho_{mk}^{(\phi)}, \tag{14}$$

which are the explicit forms of linear equations that enable us to obtain $\rho_j^{(\psi)}$ and then $\rho_{m\ell}^{(\phi)}$ $(m\neq\ell)$ from $\tau_\ell^{(\phi)}$ [17].

3. DEGENERATE MATRICES OF BIRKHOFF'S POLYTOPE

The reconstruction of Alice's state by the results of Bob's measurement in the case of arbitrary spin can be performed, in principle, in the same manner with eq. (13), but in reality, the task is nontrivial. We need to characterize all permissible matrices with positive matrix elements $\mu_{\ell,j}$, which make valid our computations and allow us to solve the system of equations (13).

First of all we have a condition

$$\sum_\ell \mu_{\ell,j} = \sum_j \mu_{\ell,j} = 1. \tag{15}$$

The matrices, which are satisfied to such conditions, are called bistochastic or doubly stochastic. The class of $N\times N$ bistochastic matrices is a $(N-1)^2$ dimensional compact convex polyhedron known as the Birkhoff's polytope $\mathcal{B}_N$ [16, 18]. The distance between two matrices is defined by

$$D(A,B) = \sqrt{\mathrm{Tr}(A-B)(A^\dagger - B^\dagger)}. \tag{16}$$

The boundary consists of corners, edges, faces, 3-faces and so on. The extreme points or corners of the polytope represent permutation matrices $P^{(N)}$.

At first let us summarize some well-known properties of two-dimensional and three-dimensional bistochastic matrices.

In the case of $N=2$, $\mathcal{B}_2$ is a line segment with the endpoints corresponding to permutation matrices

$$P_0^{(2)} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad P_1^{(2)} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{17}$$

The distance between these endpoints is equal to $D(P_0^{(2)}, P_1^{(2)})=2$. Any bisochastic matrix $\mu^{(2)}$ inside that line can be formed by combination

$$\mu^{(2)} = p_0 P_0^{(2)} + p_1 P_1^{(2)}, \tag{18}$$

with conditions

$$p_0 + p_1 = 1, \qquad 0\leq p_0\leq 1, \qquad 0\leq p_1\leq 1. \tag{19}$$

If we use a parametrization $p_0 = \cos^2\frac{\theta}{2}$, $p_1 = \sin^2\frac{\theta}{2}$, where $0\leq\theta\leq\pi$, then obtain

$$\mu^{(2)} = \begin{pmatrix}\cos^2\frac{\theta}{2} & \sin^2\frac{\theta}{2}\\ \sin^2\frac{\theta}{2} & \cos^2\frac{\theta}{2}\end{pmatrix}. \tag{20}$$

In the case of $N=3$ the Birkhoff's polytope contains 6 corners of permutation matrices

$$\begin{aligned} P_0^{(3)} &= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, P_1^{(3)} = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, P_2^{(3)} = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \\ P_3^{(3)} &= \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, P_4^{(3)} = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, P_5^{(3)} = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}. \end{aligned} \tag{21}$$

It is easy to check that there are 6 longer edges with lengths $D(P_0^{(3)}, P_3^{(3)}) = D(P_0^{(3)}, P_4^{(3)}) = D(P_3^{(3)}, P_4^{(3)}) = D(P_1^{(3)}, P_2^{(3)}) = D(P_1^{(3)}, P_5^{(3)}) = D(P_2^{(3)}, P_5^{(3)}) = \sqrt{6}$, which form two equilateral triangles placed in two orthogonal 2-planes. The other 9 edges are shorter and have a length 2. An arbitrary bistochastic matrix inside $\mathcal{B}_3$ can be represented by

$$\mu^{(3)} = \sum_{i=0}^{5} p_i P_i^{(3)}, \tag{22}$$

with condition

$$\sum_{i=0}^{5} p_i = 1 \quad (0\leq p_i\leq 1). \tag{23}$$

For any matrix $\mu^{(3)}$ the representation (22) is not unique as the dimension of the space of 3×3 bistochastic matrices is 4 and we have 6 parameters connected with the condition (23). Though the different points inside Birkhoff's polytope can correspond to the same bistochastic matrix, the representation (22) is convenient and useful to characterize a space of bistochastic matrices.

The second condition for reconstructing an initial information after quantum measurement is the existence of unitary matrix, which constricts our set, but does not decrease a dimension. Bistochastic matrices, which can be represented by $\{\mu_{\ell,j} = |\langle\phi_\ell|\psi_j\rangle|^2\}$ are called unistochastic.

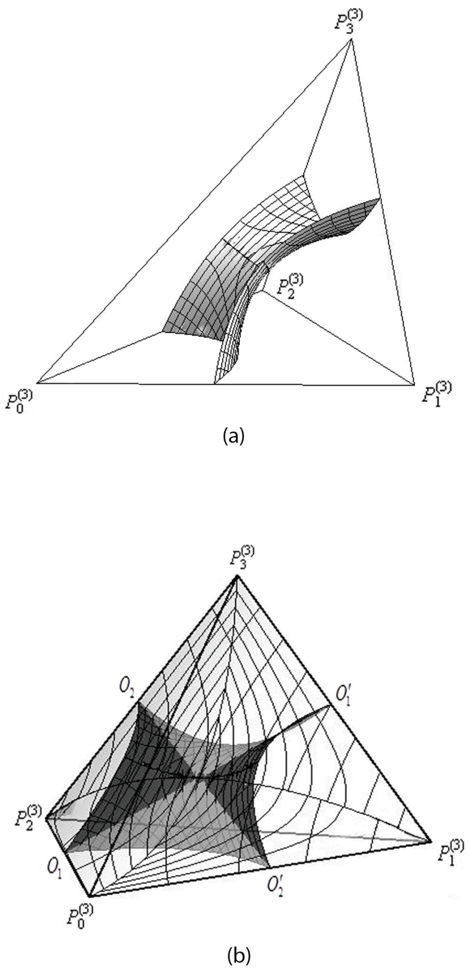

Figure 1: The surface of degenerate matrices of the 3-plain $P_0^{(3)}P_1^{(3)}P_2^{(3)}P_3^{(3)}$ of the Bikhoff's polytope (a) and it's intersection with the unistochastic surface (b). The intersection consists of two lines O_1O_1' and O_2O_2', connecting the centers of edges of irregular tetrahedron.

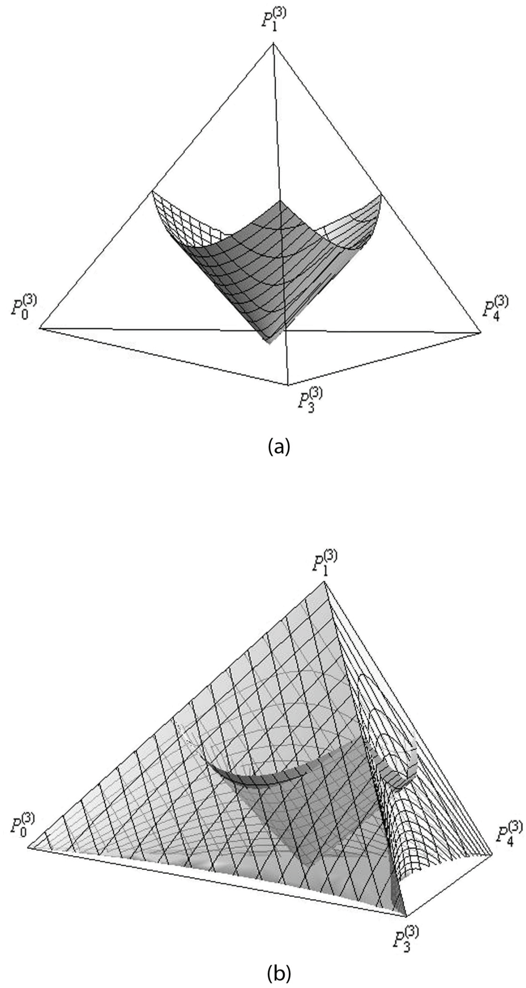

Figure 2: The surface of degenerate matrices of the facet $P_0^{(3)}P_1^{(3)}P_3^{(3)}P_4^{(3)}$ and it's intersection with the 3-surface of unistochastic matrices. The center of equilateral triangle $P_1^{(3)}P_3^{(3)}P_4^{(3)}$ belongs to the surface degenerate matrices. The boundary of unistocastic matrices on the plain $P_1^{(3)}P_3^{(3)}P_4^{(3)}$ is a 3-hypocycloid.

In a general case for arbitrary N there is no certain way to check whether the given bistochastic matrix is unis-tochastic or not. However, for $N = 2$ the answer is obvious, since all 2×2 bistochastic matrices (20) are unistochastic. For the case $N = 3$ it is always pos-sible to check whether the given bistochastic matrix is unistochastic or not. Introducing new notations

$$L_1 = \sqrt{\mu_{11}^{(3)}\mu_{12}^{(3)}},\ L_2 = \sqrt{\mu_{21}^{(3)}\mu_{22}^{(3)}},$$
$$L_3 = \sqrt{\mu_{31}^{(3)}\mu_{32}^{(3)}}, \tag{24}$$

we verify a condition of forming triangle with side lengths L_1, L_2 and L_3

$$|L_2 - L_3| \leq L_1 \leq L_2 + L_3. \tag{25}$$

If the inequalities (25) are satisfied, then the matrix $\mu^{(3)}$ is unistochastic. The unistochastic subset $\mathcal{U}_3$ of $\mathcal{B}_3$ was studied in [19]. The third condition for obtaining coefficients P_j is the existence of unique solution of the system of linear equations (13), which means that the matrix μ has to be invertible.

For the case $N = 2$ the matrix (20) is degenerated only if $\theta = \pi/2$, which corresponds to the midpoint of the segment of bistochastic matrices. This amounts to the reconfirmation of the argument of irreversibility in the previous section. When $N = 3$ we have a three-dimensional surface of degenerate bistochastic matrices, which is specified by the condition

$$\det \mu^{(3)} = 0. \tag{26}$$

Notice that the center of Birkhoff's polytope $\mu_{ij}^{(3)} = \frac{1}{3}$ also belongs to the surface of degeneracy. To characterize this surface we depict its boundaries in corresponding three-dimensional surfaces of Birkhoff's polytope. Figure (1a) illustrates a surface of degenerate bistochastic matrices, which have components $p_4 = 0$ and $p_5 = 0$ in the representation (22). Figure (1b) demonstrates an intersection of the surfaces of degenerate and unistochastic matrices. This intersection consists of two lines O_1O_1' and O_2O_2', where O_1, O_1', O_2, O_2' are the midpoints of the edges $P_0^{(3)}P_2^{(3)}$, $P_1^{(3)}P_3^{(3)}$, $P_2^{(3)}P_3^{(3)}$, $P_0^{(3)}P_1^{(3)}$ correspondingly. Thus, to obtain a set of permissible matrices $\mu^{(3)}$, we have to subtract the lines O_1O_1' and O_2O_2' from the surface of unistochastic matrices. In figure (2a) the surface of non-invertible matrices is shown within the 3-plain $P_0^{(3)}P_1^{(3)}P_3^{(3)}P_4^{(3)}$. It touches a plain $P_0^{(3)}P_3^{(3)}P_4^{(3)}$ at the center of the equilateral triangle. Note that the centers of segments $P_0^{(3)}P_1^{(3)}$, $P_3^{(3)}P_1^{(3)}$, $P_4^{(3)}P_1^{(3)}$ also belong to this surface. The set of unistochastic matrices with components $p_2 = 0$ and $p_5 = 0$ represent a three-dimensional volume, which contains edges $P_0^{(3)}P_1^{(3)}$, $P_3^{(3)}P_1^{(3)}$, $P_4^{(3)}P_1^{(3)}$. On the plain $P_0^{(3)}P_3^{(3)}P_4^{(3)}$ the boundary of unistochastic subset is the famous hypocycloid [20]. The intersection of the sets of unistochastic and degenerate matrices is shown in figure (2b). Here also the set of permissible matrices $\mu^{(3)}$ can be obtained by subtracting the set of non-invertible matrices from the volume of unistochastic ones.

4. CONCLUSION

In this article we have shown that the weak values emerge quite naturally from the gauge invariant expansion of Hermitian operators using two sets of orthonormal bases. The absence of the smooth single orthonormal basis in the limit $\{\phi_\ell\} \to \{\psi_\ell\}$ of the expansion seems to explain the reason why the concept of the weak value has eluded the discovery by all practitioners of quantum mechanics until late twenty-eighties.

It will be both very interesting mathematically and useful experimentally to characterize the unistochastic matrices of higher dimension and their irreversible subsets within the Birkhoff's polytope. It appears, however, that we have no general recipe for this task at this point, since characterizing the structure of the Birkhoff's polytope itself is already a hard task, partially completed only up to $N = 4$ [19].

5. ACKNOWLEDGEMENT

We thank Prof. I. Tsutsui, Prof. A. Tanaka, Dr. N. Yonezawa, and Dr. Shikano for stimulating discussions. This research was supported by the Japan Ministry of Education, Culture, Sports, Science and Technology under the Grant number 24540412.

REFERENCES

[1] Y. Aharonov, D. Z. Albert and L. Vaidman, "How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100", *Phys. Rev. Lett.* **60**, 1351 (1988).

[2] Y. Aharonov and L. Vaidman, "Properties of a quantum system during the time interval between two measurements", *Phys. Rev. A* **41**, 11 (1990).

[3] N. W. M. Ritchie, J. G. Story, and R. G. Hulet, "Realization of a measurement of a weak value", *Phys. Rev. Lett.* **66**, 1107 (1991).

[4] B. Reznik and Y. Aharonov, "Time-symmetric formulation of quantum mechanics", *Phys. Rev. A* **52**, 2538 (1995).

[5] Y. Aharonov and A. Botero, "Quantum averages of weak values", *Phys. Rev. A* **72**, 052111 (2005).

[6] Y. Aharonov, P. G. Bergmann and L. Lebowitz, "Time symmetry in the quantum process of measurement", *Phys. Rev.* **134**, B1410 (1964).

[7] Y. Aharonov, S. Popescu, J. Tollaksen and L. Vaidman, "Multiple-time states and multiple-time measurements in quantum mechanics", *Phys. Rev. A* **79**, 052110 (2009).

[8] A. Botero and B. Reznik, "Quantum communication protocol employing weak measurements", *Phys. Rev. A* **61**, 050301(R) (2000).

[9] Y. Aharonov, A. Botero, S. Popescu and B. Reznik, "Revisiting Hardy's paradox: counterfactual statements, real measurements, entanglement and weak values, *Phys. Lett. A* **301**, 130 (2002).

[10] E. Haapasalo, P. Lahti and J. Schultz, "Weak versus approximate values in quantum state determination", *Phys. Rev. A* **84**, 052107 (2011).

[11] J. Fischbach and M. Freyberger, "Quantum optical reconstruction scheme using weak values", *Phys. Rev. A* **86**, 052110 (2012).

[12] H. F. Hofmann, "How weak values emerge in joint measurements on cloned quantum systems", *Phys. Rev. Lett.* **109**, 020408 (2012).

[13] A. Tanaka, "Semiclassical theory of weak values", *Phys. Lett. A* **297**, 307 (2002).

[14] Y. Shikano and A. Hosoya, "Weak values with decoherence", *J. Phys. A: Math. Theor.* **43**, 025304 (2010).

[15] T. Morita, T. Sasaki and I. Tsutsui, "Complex probability measure and Aharonovs weak value", *Prog. Theor. Exp. Phys.* **2013**, 053A02 (2013).

[16] G. Birkhoff, "Three observations on linear algebra", *Univ. Nac. Tucumán Rev. A* **5**, 147 (1946).

[17] M. A. Nielsenand I. L. Chuang, "Quantum computation and quantum information", *Cambridge U.P.*, (2000).

[18] R. A. Brualdi and P. M. Gibson, "Convex polyhedra of doubly stochastic matrices. I. Applications of the permanent function", *J. Comb. Theory A* **22**, 194 (1977).

[19] I. Bengtson, Å. Ericsson, M. Kuś, W. Tadej and K. Życzkowski, "Birkhoffs polytope and unistochastic matrices, N=3 and N=4", *Commun. Math. Phys.* **259**, 307 (2005).

[20] K. Życzkowski, M. Kuś, W. Słomczynski and H.-J. Sommers, "Random unistochastic matrices", *J. Phys. A* **36**, 3425 (2003).