Parallel Algorithms of Numerical Solution of One Dynamic Problem for Quasilinear System of Equations of Elasticity Theory

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ABSTRACT

The mixed problem with first order boundary conditions for system of differential equations, which describes dynamics of homogeneous and isotropic elastic body in case of flat deformation is considered.

The numerical solution of this problem, as a rule, requires essential computing resources. One of the methods of abbreviation of time of the solution is the use of parallel algorithms. In this paper for solving of stated problem is constructed three-layer factorized difference scheme.

For the numerical solution of the received difference equations the algorithm, which can be used effectively for parallel computing systems, is offered. The pseudocode of this algorithm is given.

Keywords

Parallel algorithms, factorized difference schemes, stability, convergence.

1. STATEMENT OF PROBLEM

Let

$$D_2 = (0,\ell_1)\times(0,\ell_2) = \\ = \{x = (x_1,x_2),\quad 0 < x_1 < \ell_1, 0 < x_2 < \ell_2\}$$

be a rectangle, Γ - its boundary. $\overline{D_2}\;=D_2\;\Gamma\cup$;

$G_T\;=D_2\,(\times 0,T]$ is a cylinder in 3-dimensional Euclid space; (x, t) is any point in G_T; $\overline{G_T} = \overline{D}_2\times[0,T]$.

In $\overline{G_T}$ we considered the system of equations, which describes the dynamics of homogeneous and isotropic elastic body in case of flat deformation and has the form

$$\frac{\partial^2 u}{\partial t^2} = \mu\Delta u + (\lambda+\mu)\operatorname{grad}\operatorname{div} u + f(x,t,u), \qquad (1)$$

where $\lambda > 0$, $\mu > 0$ are Lamé's parameters, $u = \left(u^{(1)},u^{(2)}\right)$, $f = \left(f^{(1)}, f^{(2)}\right)$.

Let the function $f(x,t,u)$ satisfy the Lipschitz condition

$$\left\| f\left(x,t,u^{(1)}\right) - f\left(x,t,u^{(2)}\right)\right\| \le L\left\|u^{(1)} - u^{(2)}\right\|,$$

where $L > 0$ - is defined constant, boundary and initial conditions

$$u(x',t) = g(x',t), \quad (x',t)\in\Gamma\times[0,T] \qquad (2)$$

$$u(x,0) = u_0(x), \quad \frac{\partial u(x,0)}{\partial t} = u_1(x), \quad x\in\overline{D}_2, \qquad (3)$$

where $u_0 = \left(u_0^{(1)},u_0^{(2)}\right)$, $u_1 = \left(u_1^{(1)},u_1^{(2)}\right)$.

Rewrite the equation (1) in the following way [1]:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial x_\alpha}\left(K_{\alpha\beta}\frac{du}{dx_\beta}\right) + f(x,t,u), \qquad (4)$$

where $K_{\alpha\beta} = \begin{pmatrix} K_{\alpha\beta}^{11} & K_{\alpha\beta}^{12} \\ K_{\alpha\beta}^{21} & K_{\alpha\beta}^{22}\end{pmatrix}$, $\alpha = 1,2$,

$$K_{\alpha\beta}^{sm} = \mu\delta_{\alpha\beta}\delta_{sm} + (\lambda+\mu)\left[\theta\delta_{\alpha s}\delta_{\beta m} + (1+\theta)\delta_{\alpha m}\delta_{s\beta}\right],$$

θ - any constant, δ_{ij} - Kronecker symbol.

2. THE DIFFERENCE SCHEME

We will use designations from [2]. In the cylinder $\overline{G_T} = \overline{D}_2\times[0,T]$ let us introduce the grid. The construction of the spatial-time grid in $\overline{G_T}$ is carried out by means of the one-dimensional grids on the intervals $[0,\ell_\alpha]$, $\alpha = 1,2$, and $[0,T]$:

$$\overline{\omega}_\alpha = \left\{x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha,\quad i_\alpha = 0,1,\ldots,N_\alpha,\quad N_\alpha h_\alpha = \ell_\alpha\right\},$$

$$\omega_\alpha = \left\{x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha,\quad i_\alpha = 1,\ldots,N_\alpha - 1,\quad N_\alpha h_\alpha = \ell_\alpha\right\},$$

$$= \;\alpha 1,2\,.$$

The difference grid in parallelepiped $\overline{D}_p$ is constructed in the following way:

$$\overline{\omega}_h = \prod_{\alpha=1}^{2}\overline{\omega}_\alpha = \\ = \left\{x = (i_1h_1, i_2h_2)\in\overline{D}_2,\quad i_\alpha = 0,1,\ldots,N_\alpha,\ N_\alpha h_\alpha = \ell_\alpha\right\}.$$

where $\omega_h = \prod_{\alpha=1}^{2} \omega_\alpha$ is the set of internal points of the difference grid in D_2.

We denote by $\Gamma_h = \overline{\omega}_h \setminus \omega_h \equiv \{x \mid x \in \Gamma_h\}$ the set of knots, belonging to the boundary Γ, which are called the boundary knots. On the interval $[0, T]$ we introduce the difference grid

$$\overline{\omega}_\tau = \{t_j = j\tau, \quad j = 0,1,\dots,k, \quad k\tau = T\}.$$

Let $\Omega_{h\tau} = \omega_h \times \overline{\omega}_\tau$ be the partial-time grid of internal knots of cylinder G_T and (x, t_j) is any knot of $\Omega_{h\tau}$. $\Gamma_{h\tau} = \Gamma_h \times \overline{\omega}_\tau$ be the set of knots of difference grid on lateral surface of the cylinder $\overline{G}_T$ and (x', t_j) is knot, belonging to $\Gamma_{h\tau}$. Then

$$\overline{\Omega}_{h\tau} = \Omega_{h\tau} \cup \Gamma_{h\tau} = \overline{\omega}_h \times \overline{\omega}_\tau$$

is the partial-time grid in $\overline{G}_T$.

For problem (4), (2), (3) let us consider a three-layer difference scheme, the order of approximation of which in a class $C^{4,4}(\overline{G}_T)$ of solutions (4) is a quantity $O(\tau^2 + h^2)$:

$$y_{\bar{t}t} + \tau^2 R y_{\bar{t}t} + Ay = f(x,t,y), \quad (x,t) \in \Omega_{h\tau}, \qquad (5)$$

where

$$Ay = -\frac{1}{2} \sum_{\alpha,\beta=1}^{2} \left[\left(K_{\alpha\beta} y_{\bar{x}_\beta}\right)_{x_\alpha} + \left(K_{\alpha\beta} y_{x_\beta}\right)_{\bar{x}_\alpha} \right],$$

R is the linear operator-regularizer, whose choice provides absolute stability of difference scheme with respect to initial data and the right side, $y = \left(y^{(1)}, y^{(2)}\right)$

In [1] is proved the validity of the following relationships

$$\nu_1 A^0 \le A \le \nu_2 A^0,$$

where $A^0 y = -\left(y_{\bar{x}_1 x_1} + y_{\bar{x}_2 x_2}\right)$, $\nu_1 = \mu$, $\nu_2 = \lambda + 2\mu$.

Based on the result from [3] one can prove, that the difference scheme (5) with regularizor

$$R = \sigma A^0, \quad \sigma \ge \frac{1+\varepsilon}{4} \nu_2 \quad (0 < \varepsilon \le 1) \qquad (6)$$

is absolutely stable with respect to initial data and the right side.

3. THE FACTORIZED DIFFERENCE SCHEME

Proceeding with the construction of the factorized difference scheme, considering the scheme (5) as the initial one, we will write difference scheme (5) with regularizor (6) as follows:

$$\left(E + \tau^2 \sum_{\alpha=1}^{2} R_\alpha^0\right) y_{\bar{t}t} + Ay = f(x,t,y), \qquad (7)$$

where $R_\alpha = \sigma A_\alpha^0$, $A_\alpha^0 y = -y_{\bar{x}_\alpha x_\alpha}$, $\alpha = 1,2$.

Replacing the operator $E + \tau^2 \sum_{\alpha=1}^{2} R_\alpha^0$ with the factorized operator $\prod_{\alpha=1}^{2} \left(E + \tau^2 R_\alpha^0\right)$, we receive the factorized difference scheme

$$\prod_{\alpha=1}^{2} \left(E + \tau^2 R_\alpha^0\right) y_{\bar{t}t} = f(x,t,y) - Ay. \qquad (8)$$

Vector-function y satisfies the following initial and boundary conditions:

$$y(x',t) = g(x',t), \quad when \ (x',t) \in \Gamma_h \times \omega_\tau, \qquad (9)$$

$$y(x,0) = u_0(x), \quad y(x,\tau) = \tilde{u}_1(x), \ where \ \tilde{u}_1(x) = u_0(x) + \tau u_1(x). \qquad (10)$$

Theorem. *Let the problem* (1)-(3) *have the unique solution* $u(x,t) \in C^{4,4}(\overline{G}_T)$. *Then the solution of difference problem* (8)-(10) *converges in norm* $\overset{\circ}{W}{}_2^{(1)}$ *with speed* $O(\tau^2 + h^2)$ *to the solution of problem* (1)-(3).

4. ALGORITHM AND PSEUDOCODE OF ITS PARALLEL REALIZATION

4.1. On the base of difference scheme (8) we can construct an effective computing algorithm for parallel computing systems. Let's rewrite the difference scheme (8) in the following way:

$$S_1 S_2 y_{\bar{t}t} = F,$$

that is equivalent

$$\begin{cases} S_1 \nu^{(1)} = F, \quad S_2 \nu^{(2)} = \nu^{(1)}, \\ \nu^{(2)} = y_{\bar{t}t}, \ or \ y^{n+1} = \tau^2 \nu^{(2)} + 2y^n - y^{n-1}, \quad n = 1,2,\dots \end{cases} \qquad (11)$$

where

$$\nu^{(1)} = \left(\vec{\nu}_1^{(1)}, \vec{\nu}_2^{(1)}\right), \quad \nu^{(2)} = \left(\vec{\nu}_1^{(2)}, \vec{\nu}_2^{(2)}\right),$$

$$y = \left(y^{(1)}, y^{(2)}\right), \quad y_{\bar{t}t} = \left((\vec{y}_1)_{\bar{t}t}, (\vec{y}_2)_{\bar{t}t}\right),$$

$$F = \left(\vec{F}_1, \vec{F}_2\right), \quad F = f - Ay,$$

$$S_i = E - \tau\overline{\sigma}\Lambda_i^0, \text{ where } \Lambda_i^0 y = -y_{\bar{x}_i x_i}, \quad i = 1,2.$$

In order to define the function $\nu_{(1)}$ at $(x',t) \in \Gamma_h \times [0,T]$ we use the following formula:

$$\nu_{(1)} = S_2 g_{\bar{t}t}.$$

This algorithm is particularly suitable for solving the considered problem in the case, when the function $g(x,t)$ does not depend on t. In this case we have the homogeneous boundary conditions for functions $\nu_{(1)}$.

The first equation (11) in expanded form for $m = 1,2; \quad j = 1,2,\dots,N_2 - 1$, can be written as follows:

$$\tau\bar{\sigma}\nu_m^{(1)}\left((i-1)h, jh, t^{n+1}\right)-\left(h^2+2\tau\bar{\sigma}\right)\nu_m^{(1)}\left(ih, jh, t^{n+1}\right)+$$
$$+\tau\bar{\sigma}\nu_m^{(1)}\left((i+1)h, jh, t^{n+1}\right)=-h^2F_m\left(ih, jh, t^n\right), \quad (12)$$
$$i=1,2,\ldots,N_1-1,$$

with given initial and boundary conditions:

$$\nu_m^{(1)}(ih, jh, 0), \quad \nu_m^{(1)}(ih, jh, \tau) \quad \text{and}$$
$$\nu_m^{(1)}\left(0, jh, t^{n+1}\right), \quad \nu_m^{(1)}\left(N_1h, jh, t^{n+1}\right). \quad (13)$$

In the equations (12)-(13) the index j is fixed. With respect to the index i we have the linear system of equations with a three-diagonal matrix. Calculations are carried out parallely in respect of j.

Similarly, the second equation (11) in expanded form for $m=1,2;\ i=1,2,\ldots,N_1-1$, can be written as follows:

$$\tau\bar{\sigma}\nu_m^{(2)}\left(ih, (j-1)h, t^{n+1}\right)-\left(h^2+2\tau\bar{\sigma}\right)\nu_m^{(2)}\left(ih, jh, t^{n+1}\right)+$$
$$+\tau\bar{\sigma}\nu_m^{(2)}\left(ih, (j+1)h, t^{n+1}\right)=-h^2\nu_m^{(2)}\left(ih, jh, t^{n+1}\right), \quad (14)$$
$$j=1,2,\ldots,N_2-1,$$

with given initial and boundary conditions:

$$\nu_m^{(2)}(ih, jh, 0), \quad \nu_m^{(2)}(ih, jh, \tau) \quad \text{and}$$
$$\nu_m^{(2)}\left(ih, 0, t^{n+1}\right), \quad \nu_m^{(2)}\left(ih, N_2h, t^{n+1}\right). \quad (15)$$

In the equations (14)-(15) the index i is fixed. With respect to the index j we have the linear system of equations with a three-diagonal matrix. Calculations are carried out parallely in respect of i.

For solving the systems (12) and (14) with three-diagonal matrix one can apply one of the algorithms of one-dimensional double-sweep method. However, the solution of these systems can be carried out parallely, using parallel computing algorithms (see, for example, [4, 5]).

4.2. The pseudocode of parallel realization.

The pseudocode of algorithm has the following appearance:

$$N_1=(d_2^{(1)}-d_1^{(1)})/h, \quad N_2=(d_2^{(2)}-d_1^{(2)})/h,$$
$$b=\tau\bar{\sigma}, \quad a=\left(h^2+2\tau\bar{\sigma}\right), \quad c=\tau\bar{\sigma}.$$

where $x_1\in\left[d_1^{(1)}, d_2^{(1)}\right], \quad x_2\in\left[d_1^{(2)}, d_2^{(2)}\right].$

```
// input boundary conditions
do m = 1, 2
| y_m^(1)(0, 0:N_2),   y_m^(1)(N_1, 0:N_2),
| y_m^(1)(0:N_1, 0),   y_m^(1)(0:N_1, N_2),
| y_m^(2)(0, 0:N_2),   y_m^(2)(N_1, 0:N_2),
| y_m^(2)(0:N_1, 0),   y_m^(2)(0:N_1, N_2).
end do

// input initial conditions
  y_1^(1)(1:N_1-1, 1:N_2-1)
  y_2^(1)(1:N_1-1, 1:N_2-1)
// y^(1)(i, j) = (y_1^(1)(i, j), y_2^(1)(i, j)) - value of solution on
       t = nτ layer at x_1 = ih, x_2 = jh
  y_1^(2)(1:N_1-1, 1:N_2-1)
  y_2^(2)(1:N_1-1, 1:N_2-1)
// y^(2)(i, j) = (y_1^(2)(i, j), y_2^(2)(i, j)) - value of solution on
       t = (n-1)τ layer at x_1 = ih, x_2 = jh
α(2) = c/a

do i = 2 : max(N_1-2, N_2-2)
| α(i+1) = -c/(b*α(i) - a)
end do

do t = 2τ : T, step τ
  do m = 1, 2
    par do j = 1 : N_2-1   // solution of the first system
      β(2) = h^2 · F_m(h_1, jh_2, t, y)/a
      do i = 2 : N_1-2
      | β(i+1) = (-h^2 · F_m(ih_1, jh_2, t, y) - b·β(i)) / (b·α(i) - a)
      end do
      ν^(1)(N_1-1, j) = (-h^2 · F_m((N_1-1)h_1, jh_2, t, y) -
               - b·β(N_1-1))/(b·α(N_1-1) - a)
      do i = N_1-2 : 1, step (-1)
      | ν^(1)(i, j) = α(i+1)·ν^(1)(i+1, j) + β(i+1)
      end do
    end par do
    par do i = 1 : N_1-1   // solution of the second system
      β(2) = h^2 · ν^(1)(i,1)/a
      do j = 2 : N_2-1
      | β(j+1) = (-h^2 · ν^(1)(i, j) - b·β(j))/(b·α(j) - a)
      end do
      ν^(2)(i, N_2-1) = (-h^2 · ν^(1)(i, N_2-1) - b·β(N_2-1)) / (b·α(N_2-1) - a)
      do j = N_2-2 : 1, step (-1)
      | ν^(2)(i, j) = α(j+1)·ν^(2)(i, j+1) + β(j+1)
      end do
    end par do

    par do i = 1 : N_1-1
      par do j = 1 : N_2-1
      | y_m(i, j) = τ^2 · ν^(2)(i, j) + 2y_m^(1)(i, j) - y_m^(2)(i, j)
      end par do
    end par do
    y_m^(2) = y_m^(1)
    y_m^(1) = y_m
  end do
end do
```

Results of calculation are y_1 and y_2 for all (i, j).

„**par** do“ means that calculations can be carried out in parallel, „do $i = 1 : N - 1$ “ means the cycle, when i changes from 1 to N with step 1.

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