Analytical study of the scheme dispersion for coupled nonlinear equations

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ABSTRACT

In this paper, we examine additional nonlinear terms that may be added in the scheme to suppress oscillations and keep the steepness of the shock wave. The second order Lax-Wendroff scheme is used for discretization of the coupled gas dynamics equations. The differential approximation in the form of the coupled nonlinear partial differential equations is obtained for the scheme and an influence of artificial viscosity and additional nonlinear terms are studied using exact solutions of the differential approximation. The form of the nonlinear terms is defined when a smooth exact shock wave solution of the modified differential approximation exists.

Keywords

Discrete scheme, dispersion, nonlinear equation.

1. INTRODUCTION

The description of moving shocks for the gas dynamics equations or the Euler equations attracted considerable interest in the last decades [1, 2, 4, 3]. The choice of the most efficient numerical scheme for capturing the shocks requires sharp resolutions but without oscillations. The oscillations are caused by the features of the numerical schemes. Recently the method of differential approximation [1, 2, 4, 3] has been developed. This method allows us to study dispersive and dissipative features of a numerical discrete scheme by an analysis of a *continuum differential* equation called the differential approximation (DA) of the scheme. It is obtained by using a substitution of the Taylor expansions of the discrete functions into the *difference* scheme. The study of the resulting partial differential equation (PDE) is possible if the expansion is truncated at some order. The analysis of the schemes for linear equations results in a linear DA that helps to explain the postshock oscillations and suggest some improvements of the schemes including artificial viscosity and dispersion [4]. However, an analysis of a discrete scheme of a *non-linear* equation gives rise to a DA on the form of non-linear and nonintegrable PDE. Often the nonlinear PDE is very complicated and cannot be used for an analytical study. Therefore, some simplifications are needed for the analysis of the DA, thus for the analysis of the dispersion features of the scheme.

In this paper we develop an algorithm based on a suitable simplification of the DA giving rise to obtaining exact solutions. Existence of the solutions of the DA with permanent shape and velocity allows us to suggest additional artificial nonlinear terms that may be incorporated in the scheme. As a result smooth shock wave is described numerically. Isothermic Euler equations are studied, and the second-order Lax-Wendroff scheme is examined. Also it is shown how the addition of artificial nonlinear terms into the equations helps to solve numerically nonlinear coupled equations by means of the tools of the Wolfram Mathematica 9.

2. ISOTHERMIC EULER AND NAVIER-STOKES EQUATIONS

The isothermal coupled compressible Navier-Stokes equations (NSE) are

$$\rho_t + (\rho\ u)_x \ = \ 0, \tag{1}$$

$$(\rho\ u)_t + (\rho\ u^2 + a^2\ \rho)_x - \nu(\rho\ u_x)_x \ = \ 0, \tag{2}$$

where ρ is density, u is velocity, a- constant velocity of sound. These equations arise as a generalization of the Euler equations,

$$\rho_t + (\rho\ u)_x \ = \ 0, \tag{3}$$

$$(\rho\ u)_t + (\rho\ u^2 + a^2\ \rho)_x \ = \ 0, \tag{4}$$

by adding the viscosity term $\nu(\rho\ u_x)_x$. The reason of the modification of Eqs. (3), (4) is in suppression of parasitic oscillations caused by the second-order schemes used to account for the shock waves propagation. Indeed, artificial viscosity helps to avoid oscillations but simultaneously makes the shock smoother. This may be seen from the exact travelling wave solution of Eqs. (1), (2). The solutions should satisfy the following boundary conditions,

$$\rho \ \to \ \rho_{\pm\infty} \quad \text{at} \ x \ \to \ \pm\ \infty, \tag{5}$$

$$u \to 0 \ \text{at} \ x \to \infty, \ u \to u_{-\infty} \ \text{at} \ x \to -\infty. \tag{6}$$

Transformation to the phase variable $\theta \ = \ x - V \ \ t$ gives rise to the relationship between ρ and u following from Eq. (1)

$$\rho \ = \ \frac{\rho_\infty\ V}{V - u}. \tag{7}$$

Substitution of the last expression into Eq. (2) yields the equation for u whose solution is obtained by direct integration in the form

$$u \ = \ \frac{u_{-\infty}}{2}\ \left(1 - \tanh[k\ (X - V\ t)]\right), \tag{8}$$

where V and k are defined by

$$k = \frac{u_{-\infty}}{2\,\nu},\; V = \frac{u_{-\infty} + \sqrt{4a^2 + u_{-\infty}^2}}{2}. \quad (9)$$

Typical profiles for u are shown in Fig. 1 for different values of ν. One can see that the profile becomes steeper as the viscosity coefficient decreases and tends to the step-like shock profile of the Euler equations (3), (4).

3. DEVELOPMENT OF THE METHOD OF DIFFERENTIAL APPROXIMATION

The exact solutions obtained before are particular and they require specific initial conditions. More general solutions may be obtained only numerically. However, a discrete model itself may possess internal dispersive and/or dissipative properties caused by a method of discretization. It gives rise to non-physical deviations in the numerical solution. In particular, application of the famous Lax-Wendroff (LW) scheme results in par-asitic oscillations on the profile of the shock wave, see Fig.2.

The method of differential approximation [3, 4] allows us to study dispersive and dissipative features of a discrete equation by an analysis of a *differential* equation called the differential approximation (DA) of the scheme. According to the method of differential approximation the Taylor expansions of the discrete function is substituted into the *difference* scheme. However, instead of the strict continuum limit, some terms proportional to the powers of the spatial, $\triangle x$, and the temporal, $\triangle t$, steps are left. Truncating these series by some order, another differential equation appears, and it is called the DA. One can see that DA for a dicretization of a *non-linear*

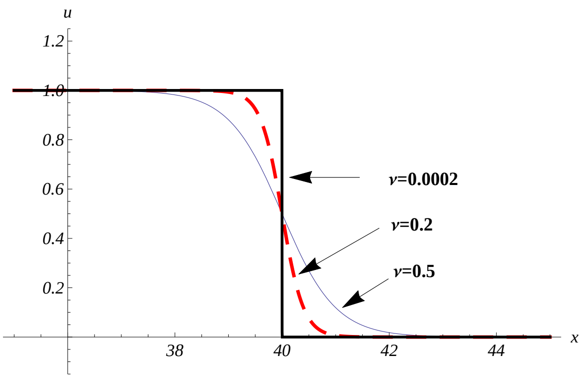

Figure 1: Typical profiles of the exact solution to the isothermal Navier-Stokes equations for various values of the kinematic viscosity coefficient.

equation is also non-linear and nonintegrable equation whose analysis is difficult. Usually the DA is written so as the l.h.s. coincides with the original equation while the r.h.s. contains all the retained terms proportional to the steps. Then a simplification may be suggested linearizing the r.h.s. parts of the DA. In our case the variables u and ρ should be linearized around their boundary values $u_{-\infty}$, $\rho_{-\infty}$. Then one obtains the simplified DAs for Eqs. (1), (2),

$$\rho_t + (\rho\, u)_x \;=\; \gamma_1 \rho_{xxx} + \gamma_2 u_{xxx}, \quad (10)$$

$$(\rho\, u)_t + (\rho\, u^2 + a^2\, \rho)_x - \nu(\rho\, u_x)_x \;=\; \gamma_3 \rho_{xxx} + \gamma_4 u_{xxx}. \quad (11)$$

with γ_i being the functions of the steps and the boundary conditions,

$$\gamma_1 \;=\; \frac{u_{-\infty}\triangle t^2}{3}\left(a^2 - u_{-\infty}^2\right),$$

$$\gamma_2 \;=\; \frac{1}{6}\left((a^2 + 3u_{-\infty}^2)\triangle t^2 - \triangle x^2\right),$$

$$\gamma_3 \;=\; \frac{1}{6}\left((a^4 - 3\,u_{-\infty}^4 + 2a^2\,u_{-\infty}^2)\triangle t^2 - (a^2 - u_{-\infty}^2)\triangle x^2\right),$$

$$\gamma_4 \;=\; \frac{u_{-\infty}}{3}\left((a^2 + u_{-\infty}^2)\triangle t^2 - \triangle x^2\right)$$

Dispersive features of the discrete LW scheme are described by the third order derivative terms, ρ_{xxx}, u_{xxx}, they may be studied using asymptotic and exact solutions of the differential equations (10), (11).

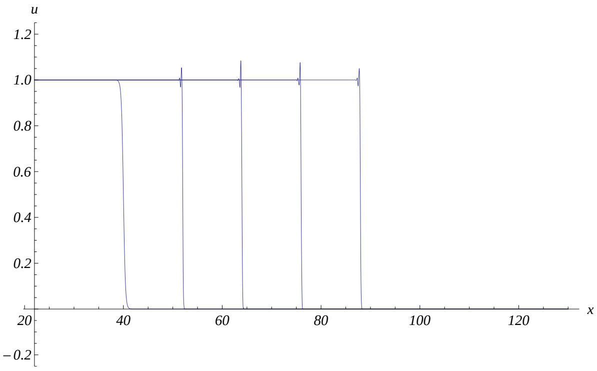

Figure 2: Deviations in shock profile of u due to the scheme dispersion.

4. EXACT SOLUTION OF THE IMPROVED DA

One can check that Eqs. (10), (11) do not possess an exact solution (7), (8). It is possible to see asymptotically that the presence of the third-order derivative terms in the r.h.s. gives rise to oscillations on the shock wave profile. To avoid oscillations, a modification is needed in the form of the additional terms in the equations that causes in turn arising of the artificial terms in the LW scheme. The exact solution (7), (8) should satisfy the improved equations. Considering traveling wave solutions, let us express ρ via u from Eq. (10). One has to note that this equation is the first DA, and the terms proportional to the square of the steps are omit-ted. Then an asymptotic representation should take it into account. Then we obtain

$$\rho = \frac{V\rho_\infty}{V-u} - \frac{1}{V-u}\left(\gamma_2\; u_{\theta\theta} + \gamma_1 \frac{\partial^2}{\partial\theta^2}\frac{V\rho_\infty}{V-u}\right)$$

Before substituting it into equation (11) we also suggest the additional terms with the coefficients α_0, α_1, β_0, β_1,

$$a^2\rho_\infty - a^2\rho + \rho u(u - V) =$$

$$\gamma_3\rho_{\theta\theta} + \gamma_4 u_{\theta\theta} + \alpha_0 u_\theta + \alpha_1 (u^2)_\theta + \beta_0\rho_\theta + \beta_1(\rho^2)_\theta. \quad (12)$$

The last equation possesses the solution (7), (8) provided that α_0, α_1, β_0, β_1, are

$$\alpha_0 \;=\; \frac{u_{-\infty}\left(\gamma_2\left(\sqrt{4a^2 + u_{-\infty}^2} - u_{-\infty}\right) + 2\gamma_4\right)}{2\nu},$$

$$\alpha_1 = -\frac{\gamma_2\left(\sqrt{4a^2+u_{-\infty}^2}-u_{-\infty}\right)+2\gamma_4}{2\nu}$$

$$\beta_0 = \frac{1}{\nu\left(\sqrt{4a^2+u_{-\infty}^2}+u_{-\infty}\right)},$$

$$\left((2a^2\gamma_1+\gamma_3 u_{-\infty})\sqrt{4a^2+u_{-\infty}^2}+\gamma_3(4a^2+u_{-\infty}^2) \right)$$

$$\beta_1 = \frac{-a^2}{\nu\rho_\infty\left(\sqrt{4a^2+u_{-\infty}^2}\left(a^2+u_{-\infty}^2\right)+u_{-\infty}\left(3a^2+u_{-\infty}^2\right)\right)}$$

$$\Big(a^2\gamma_1(\sqrt{4a^2+u_{-\infty}^2}+u_{-\infty})+$$

$$\gamma_3(u_{-\infty}\sqrt{4a^2+u_{-\infty}^2}+2a^2+u_{-\infty}^2)\Big).$$

Then a modification of the LW scheme may be suggested by adding discrete analogs of the artificial terms into the LW scheme for the NSE. It should provide an absence of a bump arising in numerical simulations keeping the shape of the solution.

5. SOLVING EQUATIONS WITH MATHEMATICA

Recent versions of initially symbolic program Mathematica got rather powerful numerical tools. The command NDSolve provides numerical solution of partial differential equations, and the method of lines is employed for that. According to this method only a spatial discretization is used while the built-in ODE solver is employed for temporal variations.

The propagation of the exact shock wave solution of Eqs. (1), (2) is confirmed by the numerical solution. Unfortunately, the command NDSolve badly works for Eqs. (1), (2) with small viscosity coefficient giving rise to the deviations in the shock shown in Fig.2. Moreover, it is sensitive to the slope of the initial condition as shown in Fig.3 by solid lines.

To overcome the problems, an improvement of the equations is suggested by adding artificial nonlinear term in Eq. (2),

$$\rho_t + (\rho\, u)_x = 0, \tag{13}$$

$$(\rho\, u)_t + (\rho\, u^2 + a^2\, \rho)_x - \nu(\rho\, u_x)_x = -\beta^*(\rho^2)_\theta. \tag{14}$$

The coefficient β^8 is supposed to be the function of the steps, the term is suggested due to the analysis of Eq. (12). One can see in Fig.3 that numerical simulation of Eqs. (13), (14) allows us to recover smooth shock wave profile and avoid deviations arising in calculations of the equations without artificial nonlinear term. The suitable value of β^* have been defined empirically trying to achieve the smoother profile.

6. CONCLUSIONS

The method of differential approximation is modified to derive the nonlinear differential equation suitable for analytical description of the defects of a scheme. The aim was to suppress the parasitic oscillations.

Recently a method of compensation of dispersion has been suggested in [5] where nonlinear artificial terms were added to avoid the LW dispersion for the nonlinear advection equation. One can see that the same technique may be employed for the coupled equations. The analytical procedure is illustrated on an example of the second- order Lax-Wendroff scheme but it may be employed for the other schemes. Further study concerns numerical confirmation of these analytical results.

One cam see that rather complicated additional artificial terms are needed in comparison with the single advection equation from [5]. However, numerical results obtained using the Mathematica program demonstrate us a possibility to add more simple artificial terms.

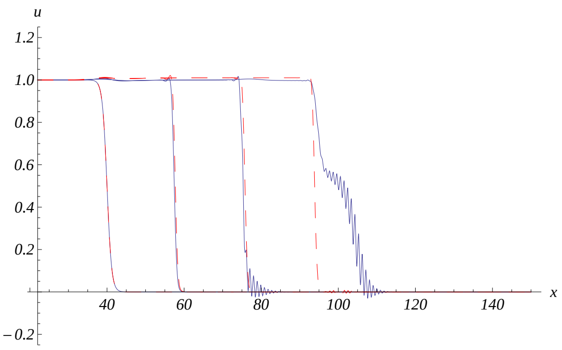

Figure 3: Deviations in shock profile of u calculated in Mathematica 9 using the NDSolve command. Shown by solid lines is the evolution of the shock calculated for Eqs. (1), (2). Dashed lines account for evolution of the shock calculated for the improved Eqs. (13), (14).

7. ACKNOWLEDGEMENT

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