On Estimation of High Degree Energy for Some Type of Hyperbolic Equations

Hakob, Abajyan
Institute for Informatics and Automation Problems
Yerevan, Armenia
e-mail: habajyan@ipia.sci.am

ABSTRACT

Under certain conditions on the coefficients $A_{ij}(t, x_1,..., x_n)$ for the solution $u(t, x_1,..., x_n)$ of the first boundary problem for the hyperbolic equations, the paper proves the estimation of high degree energy $E(t_1, v_p) \le CE(t_2, v_p)$, $t_1 \ge t_2$ with a constant C independent of t, where $v_p = \frac{\partial^p u}{\partial t^p}$.

Keywords

High degree energy, hyperbolic equations, estimation, first boundary problem.

1. INTRODUCTION

Let Ω denote the bounded domain in the space R^n with rather flat curve $\partial\Omega$. $Q = [0,\infty) \times \Omega$ – is a cylinder in the space (t, R^n), with lateral surface $S = [0,\infty) \times \partial\Omega$, $\overline{Q} = [0,\infty) \times \overline{\Omega}$ and $\overline{\Omega} = \Omega \cup \partial\Omega$.

Let $0 \le t_2 < t_1$, $Q_{t_1 t_2} = \{(x,t)$, where $x \in \Omega,\ t_2 \le t \le t_1\}$ is a cylinder with basis Ω_{t_1}, Ω_{t_2} ($\Omega_{t_j} = \{(x, t_j), x \in \Omega, j = 1,2\}$) and with lateral surface $S_{t_1 t_2} = [t_2, t_1] \times \partial\Omega$.

Let's consider the function u(t, X), which is a solution of the following problem in Q:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = Lu & (1) \\ u|_{t=0} = \varphi(x_1,..., x_n) & (2) \\ \dfrac{\partial u}{\partial t}|_{t=0} = \psi(x_1,...., x_n) & (3) \\ u|_S = 0 & (4) \end{cases}$$

where

$$Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(t, x_1,..., x_n)\frac{\partial u}{\partial x_j}\right)$$

is a symmetric elliptic operator, which means that the coefficients $A_{ij} = A_{ji}$ are functions defined in $\overline{Q}$ with some constants $0 < \gamma_1 \le \gamma_2$ and the following inequality takes place:

$$\gamma_1 \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} A_{ij}(t, x_1,..., x_n)\xi_i \xi_j \le \gamma_2 \sum_{i=1}^{n} \xi_i^2, \quad \forall (t,x) \in \overline{Q} \qquad (5)$$

We will assume that φ, ψ are chosen so that the solution of the problem (1) – (4) is sufficiently smooth.

The following expression is called energy of p-order:

$$E(t,v_p)=\int_{\Omega_t}[(\frac{\partial v_p}{\partial t})^2+\sum_{i=1}^{n}(\frac{\partial v_p}{\partial x_i})^2]d\Omega_t \ ,$$

where $v_p=\frac{\partial^p u}{\partial t^p}$, $v_0=u$, $p=0,1,...$

and u is the solution of (1) – (4).

2. THE ESTIMATION OF HIGH DEGREE ENERGY

It is well known (see [1-5]), that if $A_{ij}(t,x_1,...,x_n)$, $i,j=1,...,n$ do not depend on t, then the following theorem takes place.

Theorem 1: There always exists such a constant $c\geq 1$ so that for all $0\leq t_2\leq t_1$, the following inequality takes place: $E(t_1,v_p)\leq cE(t_2,v_p)$, $p = 0, 1....$

Now we will consider the case where $A_{ij}(t,x_1,...,x_n)$ has the following type:

$A_{ij}(t,x_1,...,x_n) = B_{ij}(x_1,...,x_n) + \alpha_{ij}(t,x_1,...,x_n)$.

In this case we have proved that the following theorem is true.

Theorem 2: If for some coefficients $c_1>0$ and $c_2>0$, the following two conditions are being satisfied:

$$\sum_{i,j=1}^{n}\sup_t\int_0^t\max_{x\in\Omega}|\frac{\partial A_{ij}}{\partial t}|d\tau < c_1 , \quad p=0, \qquad (6)$$

and

$$\sum_{i,j=1}^{n}\sum_{l=1}^{p}\sup_t\int_0^t(\max_{x\in\Omega}|\frac{\partial^l A_{ij}}{\partial t^l}|+\max_{x\in\Omega}|\frac{\partial^{l+1}A_{ij}}{\partial t^l\partial x_i}|)d\tau$$

$$< c_2 , \quad p>0, \qquad (6^*)$$

then there always exists such a constant $C>0$ so that for all $0\leq t_2\leq t_1$ for the solution of the problem (1) – (4)

$$E(t_1,v_p)\leq CE(t_2,v_p), \qquad (7)$$

where $v_p=\frac{\partial^p u}{\partial t^p}$.

3. PROPERTIES

3.1. Dependency on the initial conditions

In the set of solutions for the problem (1) – (4) let us define the following function which depends on parameter t:

$$(u,v) = \int_{\Omega_t}(u_t v_t+\sum_{i=1}^{n}u_{x_i}v_{x_i})d\Omega_t$$

It is very easy to show that it meets all the requirements of scalar product on each section Ω_t. For that it's only needed to show that the following conditions are being satisfied on each section Ω_t:

- $(u,u)\geq 0$ and $(u,u)=0 \iff u\equiv 0$
- $(u,v)=(v,u)$
- $(\alpha u,v)=\alpha(u,v)$
- $(u_1+u_2,v)=(u_1,v)+(u_2,v)$

In this case, for the solution of equation (1), the norm can be defined as the following value:

$$||u||(t) = \sqrt{(u,u)}$$

Theorem 3: The solution of the first boundary problem for equation (1) is continuously dependent on the initial conditions:

$$\forall\varepsilon>0 \quad \exists\delta=\delta(\varepsilon) \quad \|u_1-u_2\||_{t=0} < \delta \ \Rightarrow$$

$$\|u_1-u_2\||_t < \varepsilon \ \text{ in } \ \overline{Q}.$$

Proof:

$$\frac{\partial^2 u_1}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(A_{ij}(t,x_1,...,x_n)\frac{\partial u_1}{\partial x_j})=0$$

$$\frac{\partial^2 u_2}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(A_{ij}(t,x_1,...,x_n)\frac{\partial u_2}{\partial x_j})=0$$

$$u_1|_{t=0} = \varphi_1(x_1,...,x_n)$$

$$u_2|_{t=0} = \varphi_2(x_1,...,x_n)$$

$$\frac{\partial u_1}{\partial t}|_{t=0} = \psi_1(x_1,...,x_n)$$

$$\frac{\partial u_2}{\partial t}|_{t=0} = \psi_2(x_1,...,x_n)$$

$$u_1|_{S_T} = 0$$

$$u_2|_{S_T} = 0$$

Let us denote $u = u_1 - u_2$.

Since

$$\|u\|^2(t) = E(t,u) \le cE(0,u) = c$$

$$\|u_1 - u_2\|^2(0) < c\delta^2,$$

hence $\|u_1 - u_2\|(t) < \varepsilon$.

3.2. Illustrating example

Now we will show that in the Theorem of 2 conditions (6), (6 *) to some extent also are necessary, that is if do not take place (6) and (6 *) energy is not limited. For this purpose we will consider the following example.

$$u_{tt} - (2 + 4t^2)u_{xx} = 0 \qquad (8)$$

The first boundary problem for the equation (8) is like the following:

It is required to find such a function $u(t,x) \in C^2(Q_T) \cap C^1(\overline{Q_T})$, which satisfies the equation (8) in Q_T, where:

$$Q_T = (0,T) \times (0,\pi), \quad \overline{Q_T} = [0,T] \times [0,\pi]$$

and the following also takes place:

$$u|_{t=0} = \sin x \qquad (9)$$

$$\frac{\partial u}{\partial t}|_{t=0} = 0 \qquad (10)$$

initial and

$$u|_{x=0} = u|_{x=\pi} = 0 \qquad (11)$$

boundary conditions.

In the Theorem 2 we supposed that coefficients A_{ij} have the following form:

$A_{ij}(t, x_1,..., x_n) = B_{ij}(x_1,..., x_n) + \alpha_{ij}(t, x_1,..., x_n)$, i.e. $B_{ij}(x_1,..., x_n) = 2$, $\alpha_{ij}(t, x_1,..., x_n) = 4t^2$ and it is clear that in Teorema2 condition (6) does not take place because

$\int_0^t 8\tau \, d\tau = 4t^2$, i.e. the condition (6) does not take place.

It is not difficult to notice that $u(t,x) = e^{t^2} \sin x$ is the solution of equation (8).

Let us calculate the energy:

$$E(t,u) = \int_0^\pi [(\frac{\partial u}{\partial t})^2 + (\frac{\partial u}{\partial x})^2]dx = \int_0^\pi [(2te^{t^2} \sin x)^2 + (e^{t^2} \cos x)$$

$$= 4t^2 e^{2t^2} \int_0^\pi \sin^2 x dx \;\; + e^{2t^2} \int_0^\pi \cos^2 x dx =$$

$$4t^2 e^{2t^2} \int_0^\pi \frac{1 - \cos 2x}{2} dx + e^{2t^2} \int_0^\pi \frac{1 + \cos 2x}{2} dx = 2\pi t^2 e^{2t^2}$$

$$+ \frac{\pi}{2} e^{2t^2}$$

that means that energy is not limited.

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