On the *tt*-Complete Set Which is *tt*-Mitotic but not *btt*-Mitotic

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ABSTRACT

Let us adduce some definitions:
If a recursively enumerable (r.e.) set A is a disjoint union of two sets B and C, then we say that B, C is a *r.e. splitting of A*.
A r.e. set A is *tt-mitotic* (*btt-mitotic*) if there is a r.e. splitting (B,C) of A such that the sets B and C both belong to the same *tt*- (*btt*-) degree of unsolvability, as the set A.
In this paper it is proved, that there exists a *tt*-complete set, which is *tt*-mitotic, but not *btt*-mitotic.
Moreover, the constructed set A is, indeed, q-complete.

Keywords

Recursively enumerable (r.e.) set, tt -complete set, mitotic set, btt -reducibility, q -complete set

1. INTRODUCTION

Definitions and notations. We shall use the notions and terminology introduced in (Rogers [4]), (Soare [6]), (Downey and Stob [1]).

We deal with sets and functions over the nonnegative integers. $\omega = \{0,1,2,\ldots\}$.

Let us define the function $\tau(x, y)$ as follows:

$$\tau(x, y) = \frac{1}{2}\left\{ x^2 + 2xy + y^2 + 3x + y \right\}.$$

The function $\tau(x, y)$ is a 1:1 recursive function from $\omega \times \omega$ onto ω. We shall use the symbol $< x, y >$ as an abbreviation for $\tau(x, y)$.

Let π_1 and π_2 denote the inverse functions $\pi_1(< x, y >) = x$ and $\pi_2(< x, y >) = y$.

$\varphi(x)\downarrow$ denotes, that $\varphi(x)$ is defined, and $\varphi(x)\uparrow$ denotes, that $\varphi(x)$ is undefined.

The definitions of tt - and btt - reducibilities are from [4].
The ordered pair $\ll x_1,\cdots,x_k \gg, \alpha \gg$, where $< x_1,\cdots,x_k >$ is a k -tuple of integers and α is a k -ary Boolean function ($k > 0$) is called a *truth-table condition* (or tt *-condition*) *of norm* k. The set { $x_1,\cdots,x_k$ } is called the *associated set of the tt -condition*.

The tt -condition $\ll x_1,\cdots,x_k >, \alpha >$, is *satisfied* by A if $\alpha(c_A(x_1),\cdots,c_A(x_k)) = 1$, where c_A is a characteristic function for A.

Each tt -condition is a finite object; clearly an effective coding can be chosen which maps all tt -conditions (of varying norm) onto ω.

Assume henceforth that a particular such coding has been chosen. When we speak of "tt -condition x", we shall mean the tt -condition with the code number x.

Notation. *Code* $\ll x_1,\cdots,x_k >, \alpha >$ denotes the code number of tt -condition $\ll x_1,\cdots,x_k >, \alpha >$ in this coding**.**

Definitions. A is *truth-table reducible to* B (notation: $A \le_{tt} B$) if there is a recursive function f such that for all x, [$x \in A \Leftrightarrow tt$ -condition $f(x)$ is satisfied by B]. We also abbreviate "truth-table reducibility" as "tt -reducibility".

A is *bounded truth-table reducible to* B (notation: $A \le_{btt} B$), if ($\exists$recursive f) $(\exists m)(\forall x)$ [tt -condition $f(x)$ has norm $\le m$, and [$x \in A \Leftrightarrow f(x)$ is satisfied by B]].

We abbreviate "bounded truth-table reducibility" as "btt -reducibility".

Let A be the nonempty finite set $\{x_1,\cdots,x_n\}$, where $x_1 < x_2 < \cdots < x_n$. Then the integer $2^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$ is called the *canonical index of* A. If A is empty, the *canonical index* assigned to A is 0.

Let D_x be the finite set, whose canonical index is (see [4] p.70).

A is q*-reducible to* B (notation: $A \leq_q B$), if ($\exists$ recursive f) $(\forall x)\left[x \in A \Leftrightarrow D_{f(x)} B \neq \varnothing\right]$ (see [4], p.123).

A is tt*-complete* if

i) A is recursively enumerable, and

ii) $(\forall B)\left[B \text{ recursively enumerable} \Rightarrow B \leq_{tt} A\right]$.

A is q*-complete* if

i) A is recursively enumerable, and

ii) $(\forall B)\left[B \text{ recursively enumerable} \Rightarrow B \leq_q A\right]$.

2. PRELIMINARIES TO PROVE THE THEOREM

Suppose $A \leq_{tt} B$ and $(\forall x)$ [$x \in A \Leftrightarrow tt$-condition $f(x)$ is satisfied by B] and $\varphi_n = f$. Then we say that $A \leq_{tt} B$ *by* φ_n.

We say that $(A_0, A_1, \vartheta, \psi, e)$ is a *quasi-* btt *-mitotic splitting of* A if

i) (A_0, A_1) is a r.e. splitting of A and

ii) $A \leq_{btt} A_0$ by function ϑ with norm p_e (where $p_e = \pi_1(e)$) and

iii) $A \leq_{btt} A_1$ by function ψ with norm q_e (where $q_e = \pi_2(e)$).

Let us modify notations defined in (Lachlan [3]) with the purpose to adapt them to our theorem.

Definitions and notations. Let h be a recursive function from ω onto ω^5. Define $(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ to be a quintuple $(W_{e_0}, W_{e_1}, \varphi_{e_2}, \varphi_{e_3}, e_4)$, where $h(e) = (e_0, e_1, e_2, e_3, e_4)$.

If A is r.e. then we say that the *non-* btt *-mitotic condition of* e *order is satisfied for* A, if it is not the case that $(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ is a quasi- btt -mitotic splitting of A.

Let $x(e,s)$ be such a number that $\vartheta_{e,s}(x,e)\downarrow$ and $\psi_{e,s}(x,s)\downarrow$ (remind, that $\vartheta_e = \varphi_{e_2}$ and $\psi_e = \varphi_{e_3}$).

In this case as^2 denotes the associated set of tt -condition $\vartheta_e(x(e,s))$, $as^3(e,s)$ denotes the associated set of *tt*-condition $\psi_e(x(e,s))$; $as^*(e,s)$ denotes the set $as^2(e,s) \cup as^3(e,s)$.

We define a recursive function and a recursive predicate that will be of use later.

1. $L(A,e,s) = \mu n[\neg(c_A(n) = 1 \Leftrightarrow$ *tt*-condition $\vartheta_e(n)$ with norm p_{e_4} satisfied by $Y_e) \vee \neg(c_A(n) = 1 \Leftrightarrow$ *tt*-condition $\psi_e(n)$ with norm q_{e_4} satisfied by $Z_e)]$, where $h(e) = (e_0, e_1, e_2, e_3, e_4)$, $\pi_1(e_4) = p_{e_4}$, $\pi_2(e_4) = q_{e_4}$.

2. $(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ is btt *-threatening* A *through* $x(e,s)$ *at stage* s, if all the following hold:

i) $e \leq s$,

ii) $x(e,s) < L(A,e,s)$,

iii) $Y_e^s \cap Z_e^s = \varnothing$,

iv) $c_A^s(m) = (Y_e^s \cup Z_e^s)(m)$ for all $m \in as^*(e,s)$ (note that in this case $\vartheta_{e,s}(x(e,s))\downarrow$ and $\psi_{e,s}(x(e,s))\downarrow$),

v) $(\forall y \leq x(e,s))\left(\vartheta_e^s(y)\downarrow \,\&\, \psi_e^s(y)\downarrow\right) \& \left(\forall y \leq x(e,s)\right)$

[the norm of $\vartheta_e^s(y)$ is less or equal than p_{e_4} & and the norm of $\psi_e(y)$ is less or equal than q_{e_4}], where $h(e) = (e_0, e_1, e_2, e_3, e_4)$, $\pi_1(e_4) = p_{e_4}$, $\pi_2(e_4) = q_{e_4}$.

For the non- btt -mitotic condition the following proposition is true:

If $(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ is btt -threatening A through $x(e,s)$ at stage s, $x(e,s) \in A - A^s$ and for all $m \neq x(e,s)$ such that $m \in as^*(e,s)$ we have $A(m) = A^s(m)$, then the non- btt -mitotic condition of order e is satisfied for A.

This proposition is similar to Lemma 3 (about the nonmitotic condition) in (Lachlan [3]).

The satisfying method for the non- btt -mitotic condition is similar to the satisfying method for the nonmitotic condition in (Lachlan [3]).

Have a number $x(e,s)$ (so called *follower*) in the compliment of A ready to put into A if $(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ should happen to threaten A through x at some stage t and never put any other number belonging to $as^*(e,s)$ into A then.

Definitions and notations. For any set $A \subseteq \omega$ and $x \in \omega$ define the x *-column* of A. $A^{(x)} = \{<x,y>: <x,y> \in A\}$ (see Soare [5], p. 519).

$M_{y,x} = \omega^{(<y,x>)}$.

$M_e^0 = \bigcup_{i=0}^{\infty} M_{e,2i}$; $M_e^1 = \bigcup_{i=0}^{\infty} M_{e,2i+1}$.

$M^0 = \bigcup_{e=0}^{\infty} M_e^0$; $M^1 = \bigcup_{e=0}^{\infty} M_e^1$.

$\tilde{M}_{e,i} = M_{e,2i} \cup M_{e,2i+1}$; $M_e = \bigcup_{i=0}^{\infty} M_{e,i} = \bigcup_{i=0}^{\infty} \tilde{M}_{e,i}$.

Thus, $M^0 \cup M^1 = \omega$.

Let $a_0, a_1, \ldots, a_n, \ldots$ be the members of set A in increasing order. The integer a_i is denoted as $id(A)(i)$.

For any e,k define

$$M^*_{e,2k} = \left\{ id(M_{e,2k})(1), id(M_{e,2k})(2), \ldots, id(M_{e,2k})(p_{e_4} + q_{e_4} + 1) \right\},$$

$$M^*_{e,2k+1} = \left\{ id(M_{e,2k+1})(0), id(M_{e,2k+1})(1), \ldots, id(M_{e,2k+1})(p_{e_4} + q_{e_4}) \right\}$$

Let $\hat{M}_e^* = \{ id(M_{e,0})(0), id(M_{e,0})(1), \ldots,$

$id(M_{e,0})\left(\sum_{i=0}^{e} \left(p_{i_4} + q_{i_4} \right) + 1 \right) \}$.

3. PROOF OF THE THEOREM

Let us prove the following theorem.

Theorem. There exists a tt-complete set, which is tt-mitotic, but not btt-mitotic.

Proof (sketch).

It is proved using a finite injury priority argument. We construct a set A in stages s, $A = \bigcup_{s \in \omega} A_s$ such, that A is not btt-mitotic and tt-complete (moreover, the constructed set A will be, indeed, *q-complete*).

Define recursive function ξ by the following definition: $D_{\xi(x)} = \hat{M}^*_{2x}$.

We construct A to satisfy for all $e \in \omega$ the requirements:

$\tilde{R}_e$: $x \in K \Leftrightarrow \left(D_{\xi(x)} \bigcap A \right) \neq \varnothing$.

R_e : The non-*btt*-mitotic condition of order e is satisfied for A.

We also ensure that A is *tt*-mitotic.

Order the requirenments in the following priority ranking: $\tilde{R}_0, R_0, \tilde{R}_1, R_1, \tilde{R}_2, R_2, \ldots$.

R_i *requires attention at stage* s if there exists such x that $(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ is *btt-threatening A through x at stage s* and if it is not *satisfied.*

Construction

Stage $s = 0$: Let $A_0 = \varnothing$,

$x(e,0) = id(M_{e,2})(0)$ for all e.

Stage $s+1$: Act on the highest priority requirement which requires attention, if such a requirement exists:

Notation. $K = \{ x \mid x \in W_x \}$. We'll write $x \in K_{at\, s+1}$ if $x \in \left(\{ x \mid x \in W_{x,s+1} \} - \{ x \mid x \in W_{x,s} \} \right)$.

If $e \in K_{at\, s+1}$, then find number z such that $z \in \hat{M}_e^*$ & $z \notin \bigcup_{i=0}^{e} as^*(i,s)$. Such an integer z exists certainly (because $(\forall s)\left[\sum_{i=0}^{e} \left| as^*(i,s) \right| \leq \sum_{i=0}^{e} \left(p_{i_4} + q_{i_4} \right) \right]$, while $\left| \hat{M}_e^* \right| = \sum_{i=0}^{e} \left(p_{i_4} + q_{i_4} \right) + 1$). We choose least such integer z_0.

Let $z_0 = id\left(\hat{M}_{2e}^* \right)(\tilde{z})$. Then set

$A_{s+1} = A_s \bigcup \left\{ id\left(\hat{M}_{2e}^* \right)(\tilde{z}) \right\} \bigcup \left\{ id\left(\hat{M}_{2e+1}^* \right)(\tilde{z}) \right\}$.

Thus, $\tilde{R}_e$ is satisfied, declare all lower R, *unsatisfied*.

Let R_e requires attention at stage s (through $x(e,s)$).

Let $x(e,s) \in M_{e,2k}$ (that is $x(e,s) = id(M_{e,2k})(0)$).

Find z such, that ($id(M^*_{e,2k+1})(z) \notin as^*(e,s)$ & $id(M^*_{e,2k})(z) \notin as^*(e,s)$). Such an integer z exists certainly.

We choose least such integer z_0. Set

$A_{s+1} = A_s \bigcup \{ x(e,s) \} \bigcup \{ id(M^*_{e,2k})(z_0) \} \bigcup \{ id(M^*_{e,2k+1})(z_0) \}$.

Set $x(\hat{e}, s+1) = id(M_{\hat{e},2s})(0)$ for all $\hat{e} \geq e$.

Declare R_e *satisfied*, declare all lower R *unsatisfied*.

Verification

1. Since $\tilde{R}_e$ is satisfied (by the construction) for all e, so A is q-complete. Therefore, A is tt-complete.

2. Let us prove that $\tilde{A} \equiv_{tt} \tilde{\tilde{A}}$ (where $\tilde{A} = A \bigcap M^0$, $\tilde{\tilde{A}} = A \bigcap M^1$). We must construct the function g_0 which *tt*-reduces $\tilde{A}$ to $\tilde{\tilde{A}}$ and the function g_1 which *tt*-reduces $\tilde{\tilde{A}}$ to $\tilde{A}$.

In this case there would exist recursive functions $\tilde{g}_0, \tilde{g}_1$ such that $A \leq_{tt} \tilde{A}$ by function $\tilde{g}_0$ and $A \leq_{tt} \tilde{\tilde{A}}$ by function $\tilde{g}_1$, because M^0, M^1 are recursive sets.

We will construct the functions g_0, g_1 according to the following considerations.

Construction of g_0: We shall show how to compute $g_0(x)$ for any x.

There are four cases to consider:

i) If there exist e, z such that $x = id(M_{e,0})(z)$, then define $g_0(x) = code \ll id\left(M_{e,1} \right)(z) \gg, \alpha_2 \gg,$ where $\alpha_2(x) = x$ for all $x \in \{0,1\}$.

ii) If $(\neg \exists\, e,k)\ (x \in \{ id(M_{e,2k})(0) \} \bigcup M^*_{e,2k})$, then define $g_0(x) = code \ll x \gg, \alpha_0 \gg,$ where $\alpha_0(x) = 0$ for all $x \in \{0,1\}$.

iii) If $(\exists\, e)(\exists\, k \geq 1)$ ($x = id(M_{e,2k})(0)$), then define $g_0(x) = code \ll id\left(M_{e,2k+1} \right)(0), id\left(M_{e,2k+1} \right)(1), \ldots,$ $id\left(M_{e,2k+1} \right)(p+q) \gg, \alpha_1 \gg$

(where $h(e) = (e_0, e_1, e_2, e_3, e_4)$, $\pi_1(e_4) = p_{e_4}$, $\pi_2(e_4) = q_{e_4}$;

$$\alpha_1(x_0, x_1, \ldots, x_{p_{e_4}+q_{e_4}}) = \begin{cases} 0, \text{ if } x_0 = x_1 = \ldots = x_{p_{e_4}+q_{e_4}} = 0 \\ 1, \text{ otherwise.} \end{cases}$$).

iv) If $(\exists\, e)(\exists\, k > 0)$ ($x \in M^*_{e,2k}$), then define z such that $x = id\left(M^*_{e,2k} \right)(z)$.

Now define $g_0(x) = code \ll id\left(M^*_{e,2k+1} \right)(z), \gg, \alpha_2 \gg,$ where $\alpha_2(x) = x$ for all $x \in \{0,1\}$.

Construction of g_1: We shall show how to compute $g_1(x)$ for any x.

There are three cases to consider:

i) If there exist e, z such that $x = id(M_{e,1})(z)$, then define $g_1(x) = code \ll id\left(M_{e,0}\right)(z) \gg, \alpha_2 \gg,$ where $\alpha_2(x) = x$ for all $x \in \{0,1\}$.

ii) If $(\neg\exists\, e,k)\left(x \in M^*_{e,2k+1}\right)$, then define $g_1(x) = code \ll x \gg, \alpha_0 \gg,$ where $\alpha_0(x) = 0$ for all $x \in \{0,1\}$.

iii) If $(\exists\, e)(\exists k > 1)\left(x \in M^*_{e,2k+1}\right)$, then find z such, that $x = id\left(M^*_{e,2k+1}\right)(z)$.

Now define $g_1(x) = code \ll id(M^*_{e,2k})(z) \gg, \alpha_2 \gg,$ where $\alpha_2(x) = x$ for all $x \in \{0,1\}$.

The functions g_0, g_1 satisfy the abovementioned requirements.

3. We use the finite injury priority argument in the proof. So, it is following from the construction, that for each e there exits a stage s_0 such, that $(\forall s \geq s_0)\left(x(e,s_0) = x(e,s)\right)$.

For each e case a) or case b) takes place:

a) $\left(\neg\exists s \geq s_0\right)((Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ is *btt*-threatening A through $x(e,s)$ at stage s). Therefore , the non-*btt*-mitotic condition of order e is satisfied for A.

b) $\left(\exists s \geq s_0\right)\left((Y_e, Z_e, \vartheta_e, \psi_e, j_e)\right.$ is *btt*-threatening A through $x(e,s)$ at stage s).

In this case the follower $x(e,s)$ will be put into A and non-*btt*-mitotic condition of order e will be satisfied. □

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