On Incidence Colorings of Cartesian Products of Graphs

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ABSTRACT

An incidence in a graph G is a pair (v, e) with $v \in V(G)$ and $e \in E(G)$, such that v and e are incident. Two incidences (v, e) and (v', e') are adjacent if $v = v'$ or $e = e'$ or the edge vv' equals either e or e'. An incidence $k - coloring$ of G is a mapping from the set of incidences to a set of k colors such that no two adjacent incidences obtain the same color. The incidence chromatic number $\chi_i(G)$ of a graph G is the smallest k for which G has an incidence $k - coloring$. In this paper we determined the exact values of $\chi_i(P_m \square K_n)$ and $\chi_i(C_m \square K_n)$ for Cartesian products of paths and cycles with complete graphs.

Keywords

Incidence coloring, incidence chromatic number, Cartesian product, complete graph

1. INTRODUCTION

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively. The maximum degree of a graph G is denoted by $\Delta(G)$. We use standard notations P_n, C_n and K_n for simple path, simple cycle and complete graph on n vertices, respectively.

An incidence $k - coloring$ of a G is a mapping from the set of incidences to a set of k colors such that no two incidences obtain the same color. The examples of adjacent and nonadjacent incidents are pictured in Fig. 1. The incidence chromatic number $\chi_i(G)$ of a graph G is the smallest k for which G has an incidence $k - coloring$. The concept of incidence coloring of graphs was introduced by Brualdi and Massey [1] in 1993, In [1] they proved that $\Delta(G) + 1 \leq \chi_i(G) \leq 2 \cdot \Delta(G)$ for every graph G. Also in [1] the authors determined the exact value of the incidence chromatic number of trees, complete and complete bipartite graphs. In the same paper Brualdi and Massey conjectured that $\chi_i(G) \leq \Delta(G) + 2$ for every graph G (Incidence Coloring Conjecture). This conjecture was confirmed for subcubic graphs [5], outerplanar graphs [7] and square, hexagonal and honeycomb meshes [4]. However, in general, the conjecture was disproved by Guiduli [3] in 1997.

Let G and H be graphs. Define the Cartesian product $G \square H$ of graphs G and H as follows:

$$V(G \square H) = V(G) \times V(H)$$

$$E(G \square H) = \{(u_1, v_1)(u_2, v_2) \mid u_1 = u_2 \,\&\, v_1 v_2 \in E(H) \vee (v_1 = v_2 \,\&\, u_1 u_2 \in E(G))\}$$

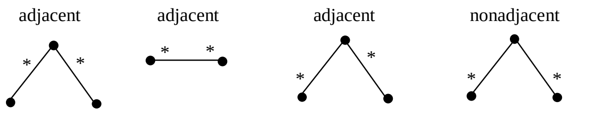

Fig. 1. Examples of adjacent and nonadjacent incidences. A * above edge e closest to vertex t represents incidence (u, e).

Incidence colorings of Cartesian products of graphs were first considered by Huang, Wang and Chang in [4]. In particular, the authors proved that $\chi_i(P_m \square P_n) = 5$ for every $m, n \geq 3$. In [2] Deming and Mingju proved that $\chi_i(P_m \square C_n) \leq \Delta(P_m \square C_n) + 2$ for every $m, n \geq 3$. In [6] Sopena and Wu showed that for every $m, n \geq 3$ $\chi_i(C_m \square C_n) = 5$ if and only if , $n \equiv 0 (mod\ 5)$; otherwise $\chi_i(C_m \square C_n) = 6$.

In the present paper we prove two results for Cartesian products of graphs, when one of the factors is a complete graph.

2. MAIN RESULTS

First we consider the Cartesian products of graphs, where one of the factors is a complete graph.

Theorem 1. For any $m, n \in N$, we have

$$\chi_i(P_m \square K_n) = \begin{cases} \Delta(P_m \square K_n) + 1, if\ m = 1\ or \\ (m = 3\ and\ n \equiv 0 (mod\ 2)) \\ \Delta(P_m \square K_n) + 2, otherwise \end{cases}$$

Thorem 2. For any $m, n \geq 3$, we have

$$\chi_i(C_m \square K_n) = \Delta(C_m \square K_n) + 2 = n + 3$$

By Theorems 1 and 2 and taking into account that a graph G with $\Delta(G) \leq 2$ contains only cycles and paths as components we obtain that the following result holds.

Corollary. If G is a graph with $\Delta(G) \leq 2$, then for any $n \geq 3$, we have $\chi_i(G \square K_n) \leq \Delta(G \square K_n) + 2$.

This corollary implies that Incidence Coloring Conjecture is true for Cartesian products of graphs when one factor is a complete graph and the second factor is a graph G with $\Delta(G) \leq 2$.

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