Interval Total Colorings of Block Graphs

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ABSTRACT

A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. An interval total $t-$ coloring of a graph G is a total coloring of G with colors $1, 2, \dots, t$ such that all colors are used and the edges incident to each vertex v together with v are colored by $d_G(v)+1$ consecutive colors, where $d_G(v)$ is the degree of a vertex v in G. A block graph is a graph in which every 2 - connected component is a clique. In this paper we prove that all block graphs are interval total colorable. We also obtain some bounds for the smallest and greatest possible number of colors in interval total colorings of these graphs.

Keywords

Total coloring, Interval total coloring, Interval coloring, Block graph

1. INTRODUCTION

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G respectively. The degree of a vertex v in G is denoted by $d_G(v)$, the maximum degree of vertices in G by $\Delta(G)$, and the total chromatic number of G by $\chi''(G)$. A block graph is a graph in which every 2 - connected component is a clique. The terms and concepts that we do not define can be found in [1,14,15].

A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. For a total coloring α of a graph G and for any $v \in V(G)$, define the set $S[v,\alpha]$ (spectrum of a vertex v) as follows: $S[v,\alpha] \equiv \{\alpha(v)\} \cup \{\alpha(e) \mid e \text{ is incident to } v\}$.

An interval total $t-$coloring [2] of a graph G is a total coloring α of G with colors $1, 2, \dots, t$ such that all colors are used and for any $v \in V(G)$, $S[v,\alpha]$ is an interval of integers. A graph G is interval total colorable if it has an interval total $t-$coloring for some positive integer t. The set of all interval total colorable graphs is denoted by $\mathfrak{T}$. For a graph $G \in \mathfrak{T}$, the smallest (the minimum span) and the greatest (the maximum span) values of t for which G has an interval $t-$coloring are denoted by $w_\tau(G)$ and $W_\tau(G)$, respectively. Clearly, $\chi''(G) \le w_\tau(G) \le W_\tau(G) \le |V(G)| + |E(G)|$ for every graph $G \in \mathfrak{T}$.

The concept of interval total coloring was introduced by Petrosyan [2]. In particular, in [2,3] the author proved that if $m+n+2-\gcd(m,n) \le t \le m+n+1$, then the complete bipartite graph $K_{m,n}$ has an interval total $t-$coloring. In [3], Petrosyan investigated interval total colorings of complete graphs and hypercubes, where he proved the following two theorems:

Theorem 1. For any $n \in \mathbb{N}$, we have

(1) $K_n \in \mathfrak{T}$,

(2) $w_\tau(K_n) = \begin{cases} n, & \text{if } n \text{ is odd}, \\ \frac{3}{2}n, & \text{if } n \text{ is even}, \end{cases}$

(3) $W_\tau(K_n) = 2n-1$.

Theorem 2. For any $n \in \mathbb{N}$, we have

(1) $Q_n \in \mathfrak{T}$,

(2) $w_\tau(Q_n) = \chi''(Q_n) = \begin{cases} n+2, & \text{if } n \le 2, \\ n+1, & \text{if } n \ge 3, \end{cases}$

(3) $W_\tau(Q_n) \ge \frac{(n+1)(n+2)}{2}$,

(4) if $w_\tau(Q_n) \le t \le \frac{(n+1)(n+2)}{2}$,

then Q_n has an interval total $t-$coloring.

Later, Petrosyan and Torosyan [12] showed that if $w_\tau(G) \le t \le W_\tau(G)$, then the complete graph K_n has an interval total $t-$coloring. In [9], Petrosyan and Shashikyan investigated interval total colorings of bipartite graphs. In particular, it was shown [8] that all trees are interval total colorable. On the other hand, in [9,10] the authors constructed examples of bipartite graphs that have no interval total coloring. Recently, Petrosyan and Khachatryan [7] proved that $W_\tau(Q_n) = \frac{(n+1)(n+2)}{2}$ for the hypercube Q_n.

Unfortunately, only a few results are known related to the problem of determining the exact values of the minimum and the maximum span in interval total colorings of graphs. The exact values of these parameters are known only for paths, cycles, trees [8,9], wheels [4], complete and complete balanced bipartite graphs [2, 3, 12]. In some papers [4-7,11,13] lower and upper bounds are found for the minimum and the maximum span in interval total colorings of certain graphs.

In the present paper we show that all block graphs are interval total colorable. Moreover, we also derive some bounds and exact values of the parameters $w_\tau(G)$ and $W_\tau(G)$ for a block graph G.

2. MAIN RESULTS

Let G be a block graph with blocks $C_1, C_2, ..., C_n$. It is well - known that all blocks are cliques. Let $E(G) = E(C_1) \cup E(C_2) \cup ... \cup E(C_n)$ and $V(G) = V(C_1) \cup V(C_2) \cup ... \cup V(C_n)$. The sequence $C_{i_1}, C_{i_2}, ..., C_{i_m}$ of blocks of a graph G, where $C_{i_p} \neq C_{i_q}, 1 \leq p < q \leq m$, is called a path of cliques $P(C_{i_1}, C_{i_m})$ of G if $V(C_{i_j}) \cap V(C_{i_{j+1}}) \neq \emptyset$ $(1 \leq j \leq m-1)$. For a path of cliques $P(C_{i_1}, C_{i_m})$ of G, define $MP(C_{i_1}, C_{i_m})$ and $LP(C_{i_1}, C_{i_m})$ as follows:

$MP(C_{i_1}, C_{i_m}) = \sum_{j=1}^{m-1} d_G\left(V(C_{i_j}) \cap V(C_{i_{j+1}})\right)$ and $LP(C_{i_1}, C_{i_m}) = m$. The path $P(C_{i_1}, C_{i_m})$ for which $MP(C_{i_1}, C_{i_m})$ has the maximum value among all possible paths of cliques of G is called the hardest path of cliques of G. Let $l_G(v)$ (v is a cut-vertex of $P(C_{i_1}, C_{i_m})$) is the number of edges which are incident to v and do not belong to $P(C_{i_1}, C_{i_m})$. For a vertex v, define $g(v)$ as follows:

$$g(v) = \max_{P_v(C_{i_p}, C_{i_q}), v \in V(C_{i_p})} MP_v(C_{i_p}, C_{i_q}).$$

Moreover, $P_v^*(C_{i_p}^*, C_{i_q}^*)$ be a path for which $g(v)$ is attained. Let v^* be a vertex such that $g(v^*) = \min_{v \in V(G)} g(v)$. Moreover, $P_{v^*}^*(C_{i_p}^*, C_{i_q}^*)$ is a path for which $g(v^*)$ is attained.

For block graphs we prove the following results:

Theorem. If G is a block graph, then

(1) $G \in \mathfrak{T}$,

(2) $$w_\tau(G) \leq \frac{3}{2}\sum_{j=p}^{q}\left|V(C_{i_j}^*)\right| - LP_{v^*}^*(C_{i_p}^*, C_{i_q}^*) + \sum_{j=p}^{q-1} l_G\left(V(C_{i_j}^*) \cap V(C_{i_{j+1}}^*)\right) + 1.$$

(3) $$W_\tau(G) = 2\sum_{j=1}^{m}\left|V(C_{i_j})\right| - 2LP(C_{i_1}, C_{i_m}) + \sum_{j=1}^{m-1} l_G\left(V(C_{i_j}) \cap V(C_{i_{j+1}})\right) + 1,$$

Our results are generalized on interval total colorings of complete graphs [3,12] and trees [8], since all these graphs are block graphs.

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