On Spanning Tree Problems Arising in Optical and Terminal Networks

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ABSTRACT

A vertex of degree one in a tree is called an end vertex and a vertex of degree at least three is called a branch vertex. For a graph G, let σ_2 be the minimum degree sum of two nonadjacent vertices in G. We consider three graph (spanning tree) problems arising in the context of optical and centralized terminal networks: finding a spanning tree of G: (i) with the minimum number of end vertices, (ii) with the minimum number of branch vertices and (iii) with the minimum degree sum of the branch vertices, motivated by network design problems where junctions are significantly more expensive than simple end- or through-nodes, and are thus to be avoided. We consider: $(*)$ connected graphs on n vertices such that $\sigma_2 \geq n-k+1$ for some positive integer k. In 1976, it was proved (by the author) that every graph satisfying $(*)$ has a spanning tree with at most k end vertices. In this paper we first show that every graph satisfying $(*)$ has a spanning tree with at most $k+1$ branch and end vertices altogether. Next, we show that every graph satisfying $(*)$ has a spanning tree with at most $(k-1)/2$ branch vertices. Finally, we show that every graph satisfying $*)$ has a spanning tree with at most $\frac{3}{2}(k-1)$ degree sum of branch vertices. All results are sharp.

Keywords

Spanning tree, end vertex, k-ended tree, branch vertex, degree sum of the branch vertices, Ore-type condition

1. INTRODUCTION

We consider only finite undirected graphs without loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$. A good reference for any undefined terms is [1].

For a graph G, we use n and α to denote the order (the number of vertices) and the independence number of G, respectively. If $\alpha \geq k$ for some integer k, let σ_k be the minimum degree sum of an independent set of k vertices; otherwise we let $\sigma_k = \infty$. We use $d(v)$ to denote the number of neighbors of a vertex v in G, called the degree of v in G. A graph G is Hamiltonian if it contains a Hamilton cycle, i.e. a cycle containing every vertex of G.

A vertex of degree one is called an end vertex (a leaf). The set of end vertices of G is denoted by $End(G)$. A branch vertex of a tree is a vertex of degree at least three. The set of branch vertices of a tree T will be denoted by $B(T)$. For a positive integer k, a tree T is said to be a $k-$ended tree if $|End(T)| \leq k$. A Hamilton path is a spanning 2-ended tree. A Hamilton cycle can be interpreted as a spanning 1-ended tree.

We begin with two famous results on Hamilton paths due to Ore [8] and Chvátal and Erdös [3].

Theorem A [8]. Every graph with $\sigma_2 \geq n-1$ has a Hamilton path.

Theorem B [3]. Every $s-$connected $(s \geq 1)$ graph with $\alpha \leq s+1$ has a Hamilton path.

A Hamilton path can be regarded as a spanning tree with maximum degree two, a spanning tree with exactly two leaves, or a spanning tree with no branch vertex. Therefore, as one of generalized problems of a Hamilton path problem, it is natural to look for conditions which ensure the existence of a spanning tree with bounded maximum degree, few leaves or few branch vertices motivated from optimization aspects with various applications.

In this paper we consider tree problems arising in the context of optical and centralized terminal networks: (i) finding a spanning tree of G with the minimum number of end vertices, (ii) finding a spanning tree with the minimum number of branch vertices and (iii) finding a spanning tree of G such that the degree sum of the branch vertices is minimized, motivated by network design problems where junctions are significantly more expensive than simple end- or through-nodes, and are thus to be avoided.

All these problems are NP-hard because they contain the Hamilton path problem as a particular case.

The constraint on the number of end vertices arises because the software and hardware associated to each terminal differs accordingly with its position in the tree. Usually, the software and hardware associated to a "degree-l" terminal is cheaper than the software and hardware used in the remaining terminals because for any intermediate terminal j one needs to check if the arrival message is destined to that node or to any other node located after node j. As a consequence, that particular terminal needs software and hardware for message routing. On the other hand, such equipment is not needed in "degree-l" terminals. Assuming that the hardware and software for message routing in the nodes is already available, the above discussion motivates a constraint stating that a tree solution has to contain exactly a certain number of "degree-l" terminals.

A different situation, resulting from a new technology allowing a switch to replicate the signal by splitting light. A light-tree connects one node to a set of other nodes in the network - allowing multicast communication from the source to a set of destinations (including the possibility of the set of destinations consisting of all other nodes). The switches which correspond to the nodes of degree greater than two have to be able to split light (except for the source of the multicast, which can transmit to any number of neighbors). Typical optical networks will have a limited number of these more sophisticated switches, and one has to position them in such a way that all possible multicasts can be performed.

Thus, we are led to the problem of finding spanning trees with as few branch vertices as possible.

2. RESULTS

In 1971, Las Vergnas [6] gave a degree condition that guarantees that any forest in G of limited size and with a limited number of end vertices can be extended to a spanning tree of G with a limited number of end vertices in an appropriate sense. This result implies as a corollary a degree sum condition for the existence of a tree with at most k leaves including Theorem A as a special case for $k = 2$.

Theorem C [2], [6], [7]. Let G be a connected graph with $\sigma_2 \geq n - k + 1$ for some positive integer k. Then G has a spanning k −ended tree.

However, Theorem C was first openly formulated and proved in 1976 by the author [7]. Later, it was reproved in 1998 by Broersma and Tuinstra [2].

Win [9] obtained a generalization of Theorem B.

Theorem D [9]. Let G be a s −connected graph with $\alpha \leq s + k - 1$ for some integer $k \geq 2$. Then G has a spanning k −ended tree.

One of the interests in the existence of spanning trees with bounded number of branch vertices arises in the realm of multicasting in optical networks.

Gargano, Hammar, Hell, Stacho and Vaccaro [5] proved the following.

Theorem E [5]. Every connected graph with $\sigma_3 \geq n - 1$ has a spanning tree with at most one branch vertex.

Flandrin et al. [4] posed the following conjecture.

Conjecture A [4]. If G is a connected graph with $\sigma_{k+3} \geq n - k$ for some positive integer k, then G has a spanning tree with at most k branch vertices.

In this note we present a sharp Ore-type condition for the existence of spanning trees in connected graphs with bounded total number of branch and end vertices. This improves Theorem C by incorporating the number of branch vertices as a parameter.

Theorem 1. Let G be a connected graph of order n. If $\sigma_2 \geq n - k + 1$ for some positive integer k, then G has a spanning tree T with at most $k - |B(T)| + 1$ end vertices.

Let G be the complete bipartite graph $K_{\delta,\delta+k-1}$ of order $n = 2\delta + k - 1$ and minimum degree δ, where $k \geq 3$. Clearly, $\sigma_2(G) = 2\delta = n - k + 1$. By Theorem 1, G has a spanning tree T with $|End(T)| \leq k - b + 1$. Observing that T is not $(k-1)$ −ended, that is $|End(T)| \geq k$, we have $b \leq 1$. On the other hand, we have $b \geq 1$, since $|End(T)| \geq k \geq 3$, which implies $b = 1$. This means that T is not $(k-b)$ −ended and consequently, Theorem 1 is sharp for each $k \geq 3$.

The next result follows from Theorem 1 providing a sharp Ore-type condition for the existence of spanning trees in connected graphs with few branch vertices.

Theorem 2. Let G be a connected graph of order n. If $\sigma_2 \geq n - k + 1$ for some positive integer k, then G has a spanning tree with at most $(k-1)/2$ branch vertices.

The third result provides an Ore-type condition for the existence of spanning trees in connected graphs with bounded degree sum of the branch vertices.

Theorem 3. Let G be a connected graph of order n. If $\sigma_2 \geq n - k + 1$ for some positive integer k, then G has a spanning tree with at most $\frac{3}{2}(k-1)$ degree sum of the branch vertices.

Let G be a graph (tree) obtained from the path $v_0 v_1 \dots v_b v_{b+1}$ by adding new vertices $u_1, \dots, u_b$ and the edges $u_i v_i$ $(i = 1, \dots, b)$. Clearly, $n = 2b + 2$ and $\sigma_2 = 2 = n - (2b + 1) + 1$. Since $B(G) = b$, the bound $(k-1)/2$ in Theorem 2 is sharp. Further, since $\sum_{i=1}^{b} d(v_i) = \frac{3}{2}(k-1)$, the bound $\frac{3}{2}(k-1)$ in Theorem 3 is sharp as well.

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