Interval Edge-Colorings with Gaps of Bipartite Graphs

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ABSTRACT

An interval (t,h)-coloring ($h \in \mathbb{Z}_+$) of a graph G is a proper edge-coloring α of G with colors $1,\dots,t$ such that all colors are used, and the colors of edges incident to each vertex v satisfy the condition $d_G(v)-1 \leq \overline{S}(v,\alpha)-\underline{S}(v,\alpha) \leq d_G(v)+h-1$, where $d_G(v)$ is the degree of a vertex v in G, $S(v,\alpha)$ is the set of colors of edges incident to v, and $\underline{S}(v,\alpha)$ and $\overline{S}(v,\alpha)$ are the smallest and largest colors of $S(v,\alpha)$, respectively. In this paper we investigate interval (t,h)-colorings of bipartite graphs. In particular, we prove that: 1) if G is a bipartite graph with $\Delta(G)=4$, then G has an interval $(4,1)$-coloring; 2) if G is a bipartite graph with $\Delta(G)=5$ and without a vertex of degree 3, then G has an interval $(5,1)$-coloring; 3) if G is a bipartite graph with $\Delta(G)=6$ and it has a 2-factor, then G has an interval $(6,1)$-coloring. We also obtain some results on interval (t,h)-colorings of biregular bipartite graphs and hypercubes.

Keywords

Interval coloring, near-interval coloring, bipartite graph, biregular bipartite graph, hypercube

1. INTRODUCTION

All graphs considered in this paper are finite, undirected, connected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of G by $\Delta(G)$, the chromatic index of G by $\chi'(G)$, and the diameter of G by $\mathrm{diam}(G)$. For $F \subseteq E(G)$, the subgraph obtained by deleting the edges of F from G is denoted by $G-F$. An (a,b)-biregular bipartite graph G is a bipartite graph G with the vertices in one part all having degree a and the vertices in the other part all having degree b. We use the standard notation Q_n for the hypercube. The terms and concepts that we do not define can be found in [3, 17].

For two positive integers a and b with $a \leq b$, the set $\{a,\dots,b\}$ is denoted by $[a,b]$ and called an interval. Let G and H be graphs. The Cartesian product $G\square H$ is defined as follows:

$$V(G\square H)=V(G)\times V(H),$$

$$E(G\square H)=\{(u_1,v_1)(u_2,v_2)\colon (u_1=u_2 \wedge v_1v_2 \in E(H)) \vee (v_1=v_2 \wedge u_1u_2 \in E(G))\}.$$

A proper edge-coloring of a graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. If α is a proper edge-coloring of G and $v \in V(G)$, then $S(v,\alpha)$ denotes the set of colors appearing on edges incident to v. The smallest and largest colors of $S(v,\alpha)$ are denoted by $\underline{S}(v,\alpha)$ and $\overline{S}(v,\alpha)$, respectively. A proper edge-coloring of a graph G is an interval t-coloring [1] if all colors are used, and for any $v \in V(G)$, the set $S(v,\alpha)$ is an interval of integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the least and the greatest values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively.

The concept of interval edge-coloring of graphs was introduced by Asratian and Kamalian [1] in 1987. In [1], they proved that if $G \in \mathfrak{N}$, then $\chi'(G)=\Delta(G)$. Asratian and Kamalian also proved [1, 2] that if a triangle-free graph G has an interval t-coloring, then $t \leq |V(G)|-1$. In [9, 10], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved that the complete bipartite graph $K_{m,n}$ has an interval t-coloring if and only if $m+n-\gcd(m,n) \leq t \leq m+n-1$, where $\gcd(m,n)$ is the greatest common divisor of m and n. In [12], Petrosyan investigated interval colorings of complete graphs and hypercubes. In particular, he proved that if $n \leq t \leq \frac{n(n+1)}{2}$, then the hypercube Q_n has an interval t-coloring. Later, in [15], it was shown that the hypercube Q_n has an interval t-coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$. In [16], Sevast'janov proved that it is an NP-complete problem to decide whether a bipartite graph has an interval coloring or not.

For subcubic bipartite graphs, Hansen proved the following

Theorem 1. [7]. If G is a bipartite graph with $\Delta(G) \leq 3$, then $G \in \mathfrak{N}$ and $w(G) \leq 4$.

For bipartite graphs with maximum degree 4, Giaro proved the following

Theorem 2. [6]. If G is a bipartite graph with $\Delta(G) =$

4 *and without a vertex of degree* 3*, then* $G \in \mathfrak{N}$ *and* $w(G) = 4$.

Let $h \in \mathbb{Z}_+$. An interval (t,h)-coloring of a graph G is a proper edge-coloring α of G with colors $1,\dots,t$ such that all colors are used, and the colors of edges incident to each vertex v satisfy the condition $d_G(v) - 1 \leq \overline{S}(v,\alpha) - \underline{S}(v,\alpha) \leq d_G(v) + h - 1$. If α is an interval (t,h)-coloring of a graph G and $v \in V(G)$, then we define $\overline{S(v,\alpha)}$ as follows:
$\overline{S(v,\alpha)} = [\underline{S}(v,\alpha), \underline{S}(v,\alpha) + d_G(v) + h - 1] \setminus S(v,\alpha)$.
A graph G is interval h-gap-colorable if it has an interval (t,h)-coloring for some positive integer t. The set of all interval h-gap-colorable graphs is denoted by $\mathfrak{N}^h$. Clearly, $\mathfrak{N}^0 \subseteq \mathfrak{N}^1 \subseteq \mathfrak{N}^2 \subseteq \cdots \subseteq \mathfrak{N}^h$. For a graph $G \in \mathfrak{N}^h$, the least and greatest values of t for which G has an interval (t,h)-coloring are denoted by $w^h(G)$ and $W^h(G)$, respectively. Note that $\mathfrak{N}^0 = \mathfrak{N}$ and $w^0(G) = w(G)$, $W^0(G) = W(G)$.

The concept of interval (t,h)-coloring of graphs was introduced by Petrosyan and Arakelyan [11] in 2007. In [11], they proved that if G is a connected graph and $G \in \mathfrak{N}^h$ ($h \in \mathbb{Z}_+$), then $W^h(G) \leq (\mathrm{diam}(G)+1)(\Delta(G)+h-1)+1$; moreover, if G is also bipartite, then $W^h(G) \leq \mathrm{diam}(G)(\Delta(G)+h-1)+1$. The authors also showed that if G is a regular graph and $h \in \mathbb{N}$, then $G \in \mathfrak{N}^h$ and $w^h(G) = \chi'(G)$; moreover, G has an interval $(t,1)$-coloring for each t satisfying $w^1(G) \leq t \leq W^1(G)$. In [4, 13], the authors determined the exact values of the parameters $w^1(G)$ and $W^1(G)$ for simple cycles, fans, wheels, complete graphs and complete bipartite graphs. Also, in [13], it was proved that all subcubic graphs are interval 1-gap-colorable. On the other hand, Petrosyan and Khachatrian [14] showed that for every $h \in \mathbb{N}$, there exists a connected graph G such that $G \notin \mathfrak{N}^h$.

In this paper interval (t,h)-colorings of bipartite graphs are investigated.

2. MAIN RESULTS

Our research is motivated by the following result.

Theorem 3. $G \in \mathfrak{N}^h$ *if and only if* $G \Box Q_h \in \mathfrak{N}$.

Proof. Let α be an interval (t,h)-coloring of G. We construct the edge-coloring β of $G \Box Q_h$ as follows: first we color each G-fibre of $G \Box Q_h$ according to α; next for each $u \in V(G)$, we color its corresponding Q_h-fibre of $G \Box Q_h$ using h colors of the set $\overline{S(u,\alpha)}$. Clearly, for each $(u,v) \in V(G \Box Q_h)$, $S((u,v),\beta) = S(u,\alpha) \cup \overline{S(u,\alpha)} = [\underline{S}(u,\alpha), \underline{S}(u,\alpha) + d_G(u) + h - 1]$. This implies that $G \Box Q_h \in \mathfrak{N}$.

Now let γ be an interval t'-coloring of $G \Box Q_h$. The restriction of this edge-coloring on the edges of the G-fibre of $G \Box Q_h$ can be transformed to an interval (t'',h)-coloring of G with $t'' \leq t'$. □

This theorem implies that each result on interval h-gap-colorability of a graph G can be transformed to the result on interval colorability of $G \Box Q_h$. Moreover, if G is a bipartite graph, then $G \Box Q_h$ is bipartite, too.

First we consider bipartite graphs with a small maximum degree. By Theorem 1, we have that if G is a bipartite graph with $\Delta(G) \leq 3$, then $G \in \mathfrak{N}^h$ and $w^h(G) \leq 4$ for every $h \in \mathbb{Z}_+$. On the other hand, the question whether all bipartite graphs with maximum degree 4 are interval colorable is an open problem [8]. Moreover, the problem remains open even for $(4,3)$-biregular bipartite graphs [8]. Nevertheless, now we show that all bipartite graphs with maximum degree 4 are interval 1-gap-colorable.

Theorem 4. *If* G *is a bipartite graph with* $\Delta(G) = 4$*, then* $G \in \mathfrak{N}^1$ *and* $w^1(G) = 4$.

Proof. Let G be a bipartite graph with maximum degree 4. If G has no vertex of degree 3, then this graph has an interval 4-coloring, by Theorem 2. Thus, $G \in \mathfrak{N}^1$ and $w^1(G) = 4$. Now we assume that G has some vertices with degree 3. Let us construct a new graph $G^\star$ as follows: we attach a pendant edge to each vertex of G with degree 3. It is easy to see that $G^\star$ is a bipartite graph with maximum degree 4 and without vertices of degree 3, so it has an interval 4-coloring, by Theorem 2 again. Now we can consider the restriction of this interval 4-coloring on the edges of the graph G. Clearly, this coloring is an interval $(4,1)$-coloring of G. □

By Theorems 1, 3 and 4, we derive the following result:

Corollary 5. *If* G *is a bipartite graph with* $\Delta(G) \leq 4$*, then* $G \Box K_2 \in \mathfrak{N}$.

In [5], the authors proved that all $(5,3)$-biregular bipartite graphs are interval 1-gap-colorable. Now we consider bipartite graphs with maximum degree 5.

Theorem 6. *If* G *is a bipartite graph with* $\Delta(G) = 5$ *and without a vertex of degree* 3*, then* $G \in \mathfrak{N}^1$ *and* $w^1(G) = 5$.

Proof. Let G be a bipartite graph with maximum degree 4 and without a vertex of degree 3. By Hall's Theorem, G has a matching that saturates all the vertices of maximum degree 5. Let M be such a matching of G. Let us consider the graph $G' = G - M$. Clearly, G' is a bipartite graph with $\Delta(G') = 4$. As in the proof of Theorem 4, we are able to prove that G' has an interval $(4,1)$-coloring α such that for each vertex $v \in V(G')$ with $d_{G'}(v) \in \{1,2,4\}$, $S(v,\alpha)$ is an interval of integers. Now we define a new edge-coloring β of G' from α by replacing colors 3 and 4 with colors 4 and 5, respectively. Clearly, for each vertex $v \in V(G')$, we have:

- if $d_{G'}(v) = 4$, then $S(v,\beta) = \{1,2,4,5\}$;
- if $d_{G'}(v) = 3$, then

$$S(v,\beta) \in \{\{1,2,4\},\{1,2,5\},\{1,4,5\},\{2,4,5\}\};$$

- if $d_{G'}(v) = 2$, then $S(v,\beta) \in \{\{1,2\},\{2,4\},\{4,5\}\}$;
- if $d_{G'}(v) = 1$, then $S(v,\beta) \in \{\{1\},\{2\},\{4\},\{5\}\}$.

Now we define an edge-coloring γ of G as follows:

1) for every $e \in E(G')$, let $\gamma(e) = \beta(e)$;

2) for every $e \in M$, let $\gamma(e) = 3$.

Since G has no vertex of degree 3, for each vertex $v \in V(G)$, we obtain:

• if $d_G(v) = 5$, then $S(v,\gamma) = [1,5]$;
• if $d_G(v) = 4$, then

$S(v,\gamma) \in \{[1,4],[2,5],\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\}\}$;

• if $d_G(v) = 2$, then

$S(v,\gamma) \in \{\{1,2\},\{2,4\},\{4,5\},\{1,3\},\{2,3\},\{3,4\},\{3,5\}\}$;

• if $d_G(v) = 1$, then $S(v,\gamma) \in \{\{1\},\{2\},\{3\},\{4\},\{5\}\}$.

This implies that γ is an interval $(5,1)$-coloring of G. □

Similarly, it can be proved that the following result holds.

Theorem 7. If G is a bipartite graph with $\Delta(G) = 5$ that has a perfect matching, then $G \in \mathfrak{N}^1$ and $w^1(G) = 5$.

In [5], the authors proved that all $(6,4)$-biregular bipartite graphs are interval 1-gap-colorable. Our next result concerns bipartite graphs with maximum degree 6.

Theorem 8. If G is a bipartite graph with $\Delta(G) = 6$ that has a 2-factor, then $G \in \mathfrak{N}^1$ and $w^1(G) = 6$.

Proof. Let G be a bipartite graph with maximum degree 6. Let F be a 2-factor of G. Let us consider the graph $G' = G - E(F)$. Clearly, G' is a bipartite graph with $\Delta(G') = 4$. As in the proof of Theorem 4, we are able to prove that G' has an interval $(4,1)$-coloring α such that for each vertex $v \in V(G')$ with $d_{G'}(v) \in \{1,2,4\}$, $S(v,\alpha)$ is an interval of integers. Now we define a new edge-coloring β of G' from α by replacing colors 3 and 4 with colors 5 and 6, respectively. Clearly, for each vertex $v \in V(G')$, we have:

• if $d_{G'}(v) = 4$, then $S(v,\beta) = \{1,2,5,6\}$;
• if $d_{G'}(v) = 3$, then

$S(v,\beta) \in \{\{1,2,5\},\{1,2,6\},\{1,5,6\},\{2,5,6\}\}$;

• if $d_{G'}(v) = 2$, then $S(v,\beta) \in \{\{1,2\},\{2,5\},\{5,6\}\}$;
• if $d_{G'}(v) = 1$, then $S(v,\beta) \in \{\{1\},\{2\},\{5\},\{6\}\}$.

Now we define an edge-coloring γ of G as follows:

1) for every $e \in E(G')$, let $\gamma(e) = \beta(e)$;

2) since F is a collection of even cycles in G, we color the edges of F alternately with colors 3 and 4.

Since G has a 2-factor, for each vertex $v \in V(G)$, we obtain:

• if $d_G(v) = 6$, then $S(v,\gamma) = [1,6]$;
• if $d_G(v) = 5$, then

$S(v,\gamma) \in \{[1,5],[2,6],\{1,2,3,4,6\},\{1,3,4,5,6\}\}$;

• if $d_G(v) = 4$, then $S(v,\gamma) \in \{[1,4],[2,5],[3,6]\}$;

• if $d_G(v) = 3$, then $S(v,\gamma) \in \{[2,4],[3,5],\{1,3,4\},\{3,4,6\}\}$;

• if $d_G(v) = 2$, then $S(v,\gamma) = \{3,4\}$.

This implies that γ is an interval $(6,1)$-coloring of G. □

Next we consider $(8,4)$-biregular bipartite graphs.

Theorem 9. If G is an $(8,4)$-biregular bipartite graph with bipartition (X,Y), then $G \in \mathfrak{N}^2$ and $w^2(G) = 8$.

Proof. Without loss of generality, we may assume that G is connected (otherwise, we consider every connected component of G). Since G is bipartite and all vertex degrees in G are even, G has a closed Eulerian trail C with an even number of edges. We color the edges of G with colors "Red"and "Blue", by traversing of the edges of G along the trail C. We color an odd-indexed edge in C with color "Red", and an even-indexed edge in C with color "Blue". Let E_R and E_B be the sets of all "Red"and "Blue"edges in G, respectively. Clearly, $E(G) = E_R \cup E_B$ and $E_R \cap E_B = \emptyset$. Define the subgraphs G_R and G_B of G as follows:

$$V(G_R) = V(G_B) = V(G) \text{ and } E(G_R) = E_R, E(G_B) = E_B.$$

Since G is Eulerian, each of subgraphs G_R and G_B of G is $(4,2)$-biregular bipartite graph with bipartition (X,Y). By the result of [7], G_R has an interval 4-coloring α such that for each $x \in X$ with $d_{G_R}(x) = 4$, $S(x,\alpha) = [1,4]$ and for each $y \in Y$ with $d_{G_R}(y) = 2$, either $S(y,\alpha) = \{1,2\}$ or $S(y,\alpha) = \{3,4\}$. We define a new edge-coloring α' of G_R from α by replacing colors 3 and 4 with colors 7 and 8, respectively. Clearly, α' is a proper edge-coloring of G_R with colors $1,2,7,8$. Moreover, for each $x \in X$ with $d_{G_R}(x) = 4$, $S(x,\alpha') = \{1,2,7,8\}$ and for each $y \in Y$ with $d_{G_R}(y) = 2$, either $S(y,\alpha') = \{1,2\}$ or $S(y,\alpha') = \{7,8\}$. Similarly, by the result of [7], G_B has an interval 4-coloring β such that for each $x \in X$ with $d_{G_B}(x) = 4$, $S(x,\beta) = [1,4]$ and for each $y \in Y$ with $d_{G_B}(y) = 2$, either $S(y,\beta) = \{1,2\}$ or $S(y,\beta) = \{3,4\}$. We define a new edge-coloring β' of G_B from β by replacing colors 1 and 2 with colors 5 and 6, respectively. Clearly, β' is a proper edge-coloring of G_B with colors $3,4,5,6$. Moreover, for each $x \in X$ with $d_{G_B}(x) = 4$, $S(x,\beta') = [3,6]$ and for each $y \in Y$ with $d_{G_B}(y) = 2$, either $S(y,\beta') = \{3,4\}$ or $S(y,\beta') = \{5,6\}$.

Finally, we define an edge-coloring γ of G as follows:

1) for every $e \in E(G_R)$, let $\gamma(e) = \alpha'(e)$;

2) for every $e \in E(G_B)$, let $\gamma(e) = \beta'(e)$.

Clearly, γ is a proper edge-coloring of G with colors $1,\ldots,8$ such that for each $x \in X$, $S(x,\gamma) = [1,8]$, and for each $y \in Y$, $S(y,\gamma) \in \{[1,4],[5,8],\{1,2,5,6\},\{3,4,7,8\}\}$.

This shows that γ is an interval $(8,2)$-coloring of G. Thus, $G \in \mathfrak{N}^2$ and $w^2(G)=8$. $\square$

Corollary 10. *If G is a $(7,4)$-biregular bipartite graph with bipartition (X,Y), then $G \in \mathfrak{N}^2$ and $w^2(G) \leq 8$.*

Proof. Clearly, $7|X| = 4|Y|$. This implies that $|X| = 4k$. Let $X = \{x_1,\ldots,x_{4k}\}$. Now we define an auxiliary graph G' as follows:

$$V(G') = X \cup Y \cup Y' \text{ and } E(G') = E(G) \cup E', \text{ where}$$
$$Y' = \{y'_1,\ldots,y'_k\}, \text{ and}$$
$$E' = \{y'_i x_{4i-3}, y'_i x_{4i-2}, y'_i x_{4i-1}, y'_i x_{4i} \colon\ 1 \leq i \leq k\}.$$

Clearly, G' is an $(8,4)$-biregular bipartite graph with bipartition $(X, Y\cup Y')$. By Theorem 9, G' has an interval $(8,2)$-coloring. It is not difficult to see that the restriction of this edge-coloring on the edges of G induces an interval $(t,2)$-coloring with $7 \leq t \leq 8$. $\square$

Finally we consider hypercubes.

Theorem 11. *If $n \in \mathbf{N}$, then*

$$\tfrac{n^2+3n-2}{2} \leq W^1(Q_n) \leq \tfrac{n^2+3n+2}{2}.$$

Proof. First of all let us note that $W^1(Q_n) \geq \frac{n^2+3n-2}{2}$ for any $n \in \mathbf{N}$, by the results of [13]. For the proof of the theorem, it suffices to show that $W^1(Q_n) \leq \frac{n^2+3n+2}{2}$ for any $n \in \mathbf{N}$. Let φ be an interval $\left(W^1(Q_n),1\right)$-coloring of Q_n.

Let $i=0$ or 1 and $Q^{(i)}_{n+1}$ be a subgraph of the graph Q_{n+1}, induced by the vertices

$\{(i,\alpha_2,\alpha_3,\ldots,\alpha_{n+1}) \colon\ (\alpha_2,\alpha_3,\ldots,\alpha_{n+1}) \in \{0,1\}^n\}$.

Clearly, $Q^{(i)}_{n+1}$ is isomorphic to Q_n for $i \in \{0,1\}$.

We define an edge-coloring ψ of Q_{n+1} as follows:

1) for $i=0,1$ and every edge $(i,\bar{\alpha})\,(i,\bar{\beta}) \in E\left(Q^{(i)}_{n+1}\right)$, let

$$\psi\left((i,\bar{\alpha})\,(i,\bar{\beta})\right) = \varphi\left(\bar{\alpha}\bar{\beta}\right);$$

2) for every $\bar{\alpha} \in \{0,1\}^n$, let

$$\psi\left((0,\bar{\alpha})\,(1,\bar{\alpha})\right) =$$
$$\begin{cases} c, \text{ where } c \in [\underline{S}((0,\bar{\alpha}),\varphi), \overline{S}((0,\bar{\alpha}),\varphi)] \setminus S((0,\bar{\alpha}),\varphi), \\ \qquad \text{if } \overline{S}((0,\bar{\alpha}),\varphi) - \underline{S}((0,\bar{\alpha}),\varphi) = n, \\ \overline{S}((0,\bar{\alpha}),\varphi)+1, \text{ if } \overline{S}((0,\bar{\alpha}),\varphi) - \underline{S}((0,\bar{\alpha}),\varphi) = n-1. \end{cases}$$

It is not difficult to see that ψ is an interval coloring of Q_{n+1} with either $W^1(Q_n)$ or $W^1(Q_n)+1$ colors. Thus, by the results of [15], we obtain $W^1(Q_n) \leq W(Q_{n+1}) = \frac{(n+1)(n+2)}{2}$ for any $n \in \mathbf{N}$. $\square$

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