Constraints and Characterization of Subsets of n-Dimensional Unit Cube

Levon Aslanyan
Institute for Informatics and Automation Problems
Yerevan, Armenia
e-mail: lasl@sci.am

Hans-Dietrich Gronau
Institut für Mathematik, Universität Rostock
Rostock, Germany
e-mail: gronau@uni-rostock.de

Hasmik Sahakyan
Institute for Informatics and Automation Problems
Yerevan, Armenia
e-mail: hsahakyan@sci.am

Peter Wagner
Institut für Mathematik, Universität Rostock
Rostock, Germany
e-mail: peter.wagner@uni-rostock.de

ABSTRACT
Characterization of n-cube subsets is considered in terms of variable frequencies (differences) or in terms of hypergraphs – the degree sequences. Characterization under the specific constraints is considered – those are the upper sets, Sperner sets and random sets. Above the general knowledge about the subset characteristics these specific classes are characterized by a set of properties – inclusion one into the other, reductions and complexity issues, example series and random descriptions.

Keywords
Set system, hypergraph, degree sequence, combinatorics.

1. INTRODUCTION
Let $X = \{x_1, \cdots, x_n\}$ be a finite set of elements. H*ypergraph* on X is a family $H = \{E_1, \cdots, E_m\}$ of subsets of X such that:

(1) $E_i \neq \emptyset, i = 1, \cdots, m$

(2) $\bigcup_{i=1}^{m} E_i = X$.

The elements of X are called *vertices*, and the sets $E_1, \cdots, E_m$ are called *hyperedges*. Hypergraph H is *simple*, if

(3) $E_i \subset E_j \Rightarrow i = j$.

The *degree* d_i of a vertex x_i is the number of hyperedges containing x_i. $(d_1, \cdots, d_n)$ is called the *degree sequence* of H. The *hypergraph degree sequence problem* is the problem of existence of a simple hypergraph by the given degree sequence. This is an open problem, first stated in [2]. Further the problem is investigated in [5, 7-13] but the complexity status is still open even for the case of 3-uniform hypergraphs (hyperedges contain exactly 3 vertices).

Hyperedges of the hypergraph H can be coded with (0,1) sequences of length n such that j-th component of the sequence equals 1 if and only if the hyperedge contains the vertex x_j. We get a binary matrix M of size $m \times n$, where the numbers of ones in rows correspond to the cardinalities of hyperedges, and the numbers of ones in columns correspond to the degrees of vertices.

$R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ denote the row and column sum vectors (the number of ones) of M. Given a binary matrix, then the values of R and S can be easily calculated. Consider the inverse problem:

given the row sum R and/or column sum S for a binary $m \times n$ matrix with a question on existence of such a matrix, or equivalently, - existence of a hypergraph with the given cardinalities of hyperedges and with the given degree sequence.

The known Gale-Ryser's theorem solves the problem in its general form, in polynomial time:

Theorem 1 [1]. Let $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ be non-increasing, positive integer vectors such that $r_i \leq n$ for $i = 1, \cdots, m$, and $s_j \leq m$ for $j = 1, \cdots, n$. Let $S^* = (s_1^*, \cdots, s_n^*)$ denote the conjugate vector of R: $s_i^* = |\{r_i : r_i \geq j,\ i = 1, \cdots, m\}|$. There exists a binary $m \times n$ matrix with row sum R and column sum S if and only if S is majorized by R^*, that is: $\sum_{i=1}^{k} s_i \leq \sum_{i=1}^{k} s_i^*$, for $k = 1, \cdots, n-1$, and $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} s_i^*$.

There exist a number of combinatorial problems, formulated in the terms of row and column sums of binary matrix M, in which additional constraints are imposed on M. *Non repetition* of rows and *incomparability* of rows are the most common constraints. These constraints originate from the existing problems of combinatorial theory: the hypergraph degree sequence problem [2]; the Boolean function variable activities problem, which is well known in its multi-dimensional binary cube description [3], as well as discrete tomography problems [4], and others.

We will formulate problems and results in the multi-dimensional binary cube terminology.

Let E^n denote the set of vertices of the n dimensional binary cube: $E^n = \{(x_1, \cdots, x_n) / x_i \in \{0,1\}, i = 1, \cdots, n\}$. $x_1, \cdots, x_n$ are the n generating variables.

The geometrical mapping is by the Hasse diagram. It has $n+1$ levels, where k-th $(k = 0, \cdots, n)$ level contains all vertices of E^n which have exactly k components equal to 1. Edges connect those vertices in neighbor levels related by a cover relation. Figure 1 shows the Hasse diagram of E^5.

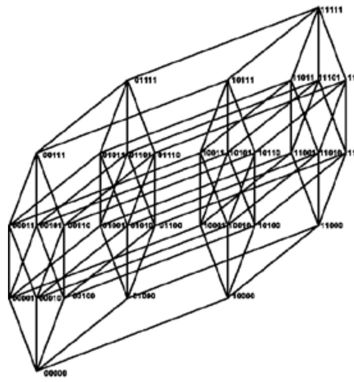

Figure1
Hasse diagram of E^5

Each subset $M \subseteq E^n$, $|M| = m$ can be identified with a binary $m \times n$ matrix M with distinct rows (we use the same letter M for a subset of vertices and for the corresponding binary matrix – this will not cause a confusion): columns correspond to the variable values, and rows correspond to the vertices of M.
Consider the partition of E^n according to the variable x_i. We get two $(n-1)$-dimensional sub-cubes of E^n:
$E^{n-1}_{x_i=1} = \{(x_1, \cdots, x_n) \in E^n: x_i = 1\}$
$E^{n-1}_{x_i=0} = \{(x_1, \cdots, x_n) \in E^n: x_i = 0\}$.
The vertex subset M will have its parts $M_{x_i=1}$ and $M_{x_i=0}$ in these sub-cubes. Observe that $|M_{x_i=1}|$ is the number of ones of the i-th column of the matrix M. The vector $(|M_{x_1=1}|, |M_{x_2=1}|, \cdots, |M_{x_n=1}|)$ - we will call the associated with M vector of partitions.
We formulate the following problems.
***P1*-** **Existence** of an m-subset of E^n by the given associated vector of partitions.
In case of existence:
***P2*-** **Reconstruction** of the-subset of E^n by the given associated vector of partitions.
***P3*-** **Characterization** of the set of all associated vectors of partitions of m-subsets of E^n.
In this paper we consider the problems ***P1, P2, P3*** for special classes of subsets of E^n, and derive particular new properties and relations for them. On the one hand the consideration of special large classes of subsets will help with the characterization, ***P3***. On the other hand, the constraints that give rise to these classes occur naturally in diverse applications. We will consider subset classes such as: sub-cubes, layers, Sperner families, monotone Boolean functions, random subsets, etc.
The paper is organized as follows: Section 2 below considers special classes of subsets and defines the corresponding problems. Complexity relations are investigated in Section 3. In Section 4 relations between subset classes are considered. Section 5 analyses the properties of random subsets of E^n.

2. SUBSET CLASSES

In this section we consider three classes of subsets in E^n, and define corresponding problems for them.

Sperner families

A family of sets (A, F) is *Sperner family* if none of the sets of F is contained in another. Equivalently, a Sperner family is an *antichain* in E^n (a subset of vertices, such that any two elements are incomparable).

Upper sets

A subset US of vertices in E^n is an *upper set* if it obeys the property: if $x \in US$ and $x \leq y$, then $y \in US$.

If we consider monotone Boolean functions defined on E^n, then upper-sets can be identified with the sets of vertices, where the functions accept the value “one”.
There is an easy relation between Sperner families and upper sets: the set of “lower units” of monotone Boolean functions compose a Sperner family, and vice versa – any Sperner family compose the set of “lower units” of some monotone Boolean function.

Random sets

Probabilistic combinatorial theory [14-16] considers discrete structures under different i.i.d. (independent and identical probability distributions). Then, parameter estimations are interpreted in terms of particular problems. E.g. analysis of all n-variable Boolean functions is equivalent to i.i.d. where variables accept values 0 and 1 with equal ½ probability. Such models are deeply investigated and applied in the test theory, in the minimization of Boolean functions, in pattern recognition theory [3]. An extension of i.i.d. when values 0 and 1 are adopted by asymmetric probabilities p and (1-p) will be considered under the point of view ***P1-P3.***
Thus, our interest is in considering:

- Arbitrary set of vertices in E^n,
- Sperner families or simple hypergraphs;
- Upper sets or monotone Boolean functions;
- Random sets in E^n.

Notations of the problems for these sets concerning ***P1, P2, P3*** - *existence*, *reconstruction*, and *characterization* - are given in Table 1 below.

	Arbitrary sets	*Sperner families*	*Upper sets*
Existence	***P1***	***P1-SF***	***P1-US***
Reconstruction	***P2***	***P2-SF***	***P2-US***
Characterization	***P3***	***P3-SF***	***P3-US***

Table 1: Problems for subset classes

We make the following notations. For a given m and n:

- $D_m(n)$ is the set of all associated vectors of partitions of arbitrary m-subsets of E^n,
- $S_m(n)$ is the set of associated vectors of m-element Sperner families in E^n.
- $U_m(n)$ is the set of associated vectors of partitions of m-element upper sets in E^n.

3. COMPLEXITY RELATIONS

In this section we consider relations, particularly, the complexity relations between the formulated problems.
We will consider also particular cases of the problems, which will help with the complexity relations.

P1-SF-k

This is a set of particular cases of ***P1-SF*** **–** where all elements of the Sperner family are k-sets. k is a generative (arbitrary) value.
Thus, ***P1-SF-k*** denote the problem of existence of an m-element Sperner family on the k-th level of E^n, given the associated vectors of partitions.

P1-US-max

For the given n and m, consider the following canonical representation form of m:

$$m = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{n-k} + m_1,\ m_1 < \binom{n}{n-k-1} \qquad (1)$$

Consider $\mathcal{M}$, - the set of the following m-element vertex subsets in E^n: take the series of all $\binom{n}{n-i}$ vertices from the $(n-i)$-th, $i = 0, \cdots, k$ levels of E^n, and the remaining m_1 vertices - from the $(n-k-1)$-th level of E^n. The choice of m_1 vertices from the $(n-k-1)$-th level is arbitrary ($\binom{n-k-1}{m_1}$ possibilities), and the elements of $\mathcal{M}$ differ from each other by these vertices only. Obviously, the elements of $\mathcal{M}$ are upper sets. We call these sets – *maximal* upper sets.
Let $\mathcal{S}$ denote the set of all associated vectors of partitions of the sets of $\mathcal{M}$. It is clear that all vectors of $\mathcal{S}$ have the same sum of components, which is equal to:
$\sum_{i=0}^{k}\left((n-i)\cdot\binom{n}{n-i}\right) + (n-k-1)\cdot m_1$, and this is the maximal possible value for associated vectors of m-sets.

P1-US-max

denote the problem of existence of an m-element maximal upper set in E^n with the given associated vector of partitions.

Complexity

P1-SF, */*and its particular cases ***P1-SF-k/*** – are equivalent to the hypergraph degree sequence problem /the case of uniform hypergraphs/ and thus, are open problems [2, 8].

Statement 1. ***P1-US-max*** and ***P1-SF-k*** are complexity equivalent problems.

First we outline the proof of ***P1-SF-k*** $\propto$ ***P1-US-max.***
Let I_1 be an instance of ***P1-SF-k*** **–** given n_1 - the size of the cube, m_1 - the size of the Sperner family, k - the size of

subsets/level of the cube, and $(q_1, \cdots, q_{n_1})$ - the given vector of partitions. We compose I_2, the instance $(n_2, m_2, (s_1, \cdots, s_{n_2}))$ of ***P1-US-max*** in the following way: $n_2 = n_1$, $m_2 = \binom{n_2}{n_2} + \binom{n_2}{n_2-1} + \cdots + \binom{n_2}{n_2-k+1} + m_1$; and $s_i = q_i + \sum_{j=0}^{k} \binom{n_2-1}{n_2-j-1}$. Obviously, I_2 is a "yes" instance of ***P1-US-max*** if and only of I_1 is a "yes" instance of ***P1-SF-k.***
The reduction ***P1-US-max*** $\propto$ ***P1-SF-k*** can be treated in a ***s***imilar way.
Given that ***P1-US-max*** is a particular case of ***P1***, we get that ***P1-SF-k***$\propto$ ***P1,*** which implies:
Statement 2. ***P1-P3*** are not easier than ***P1-SF-k.***
It is known in [10] that characterization of $D_m(n)$, the set of all associated vectors of partitions of arbitrary m-subsets of E^n, can be given through $U_m(n)$, the set of associated vectors of partitions of m-element upper sets in E^n.
It follows that:
Statement 3. ***P3-US*** is not easier than ***P3.***

Let us summarize the complexities for problems of Table 1.

Problems	*Complexity*
P1,P2,P3	*Not easier than P1-SF, P2-SF, P3-SF*
P1-SF, P2-SF, P3-SF	*Are known open problems*
P1-US	*?*
P2-US	*?*
P3-US	*Not easier than P3*

Table 2: Complexities

4. RELATIONS/PROPERTIES

In this section we consider structural relations by the set of problems ***P1-P3*** between Sperner families and upper sets in E^n.
Statement 4. Let M be an m-element upper set in E^n /the corresponding binary matrix of size $m \times n$/, and $(s_1, s_2, \cdots, s_n)$ is its associated vector of partitions. Then all columns of M with $s_i < m$ are incomparable.
Proof.

1. First we prove that there are no coinciding columns in M among those with $s_i < m$.

Suppose for the sake of contradiction that i -th and j-th columns are coinciding, and $s_i < m$ and $s_j < m$.
Consider the partitioning of E^n according to two variables: x_i and x_j . Let
$E^{n-2}_{x_i=1,x_j=1} = \{(x_1, \cdots, x_n) \in E^n: x_i = 1, x_j = 1\}$
$E^{n-2}_{x_i=1,x_j=0} = \{(x_1, \cdots, x_n) \in E^n: x_i = 1, x_j = 0\}$
$E^{n-2}_{x_i=0,x_j=1} = \{(x_1, \cdots, x_n) \in E^n: x_i = 0, x_j = 1\}$
$E^{n-2}_{x_i=0,x_j=0} = \{(x_1, \cdots, x_n) \in E^n: x_i = 0, x_j = 0\}$
denote the corresponding $(n-2)$-dimensional sub-cubes of E^n; and $M_{x_i=1,x_j=1}$, $M_{x_i=1,x_j=0}$, $M_{x_i=0,x_j=1}$, and $M_{x_i=0,x_j=0}$ denote the parts of M in these sub-cubes.
If $M_{x_i=1,x_j=0}$ and $M_{x_i=0,x_j=1}$ are empty, then $M_{x_i=0,x_j=0}$ is empty as well /by the definition of upper sets/. Hence all vertices of M occur in $E^{n-2}_{x_i=1,x_j=1}$, and therefore $s_i = s_j = m$. We get a contradiction.

2. Now we prove that none of the columns of M with $s_i < m$ contains another.

Suppose that this is not the case and the j-th column contains the i-th one. Consider the partitioning of E^n according to two variables: x_i and x_j. Then $M_{x_i=0,x_j=1}$ is empty, and hence $M_{x_i=0,x_j=0}$ is empty as well. All vertices of M occur in $E^{n-1}_{x_j=1}$ and therefore $s_j = m$. We get a contradiction, which completes the proof.

Thus, the class of upper sets is a subclass of binary matrices with distinct rows and incomparable columns.

In this regard we investigate a set of related problems of the existence of $m \times n$ binary matrices, - given in Table 3 below. The rows of the table correspond to the given conditions, and the columns contain the constraint imposed.

	Diff.rows Incomp. Columns	*Incomp. Columns*	*Incomp. Rows*
Column sum	***P1-1***	***P1-2***	***P1-SF***
Row sum	***P1-3***	***P1-4***	***P1-5***

Table 3

Future work will concern investigations on complexity relations between these problems.

5. RANDOM SET PROPERTIES

Consider random $m \times n$ Boolean matrix M where the column variables $x_j, j = \overline{1,n}$ independently and identically attain values 1 and 0 with respective none zero probabilities p_j and $q_j = 1 - p_j$. The number of generated matrices is 2^{mn} and the probability of a matrix is tightly related with the number of 1 values in its columns. Given a matrix, and column weights $s_1, s_2, \ldots, s_n$ then its probability is equal to $\prod_{j=1}^{n} p_j^{s_i} q_j^{m-s_j}$. The model is expressed as a process, where m vertices i.i.d. as it is defined above are dropped into the E^n. Vertices can appear repeatedly and in cases when there are no row repetitions we receive an m-subset of the E^n. The indicator of existence of such row different matrices can be the related nonzero probability - in the given model. We suppose that $p_j = s_j/m$ and intend to prove asymptotically, when $n, m \to \infty$ the following:

1. probability that column weights are equivalent to $\mathrm{s}_1, \mathrm{s}_2, \ldots, \mathrm{s}_\mathrm{n}$ tends to 1,
2. probability that all rows are different tends to 1.

Consider an arbitrary column j. Let M_j and D_j be the average and dispersion of the random weight of column j.

Additionally we consider the issue:

3. Probability that a random set is a SF.

Probability of weight t on the column of a random matrix equals $C_m^t p_j^{s_i} q_j^{m-s_j}$. On this base
$$M_j = \sum_{t=0}^{m} t C_m^t p_j^t q_j^{m-t} = m p_j$$

which is quite obvious. For example, when $p_j = q_j = 1/2$ we receive parameters of the usual homogeneous model of random Boolean functions on E^n. Now, the overall average weights by the set of coordinates/columns will be equal to $mp_1, mp_2, \ldots, mp_n$. And because of $p_j = \frac{s_j}{m}, j = \overline{1,n}$ we receive that the average weight vector equals to $s_1, s_2, \ldots, s_n$.
In a similar way the expected number of different pairs of rows in the one-column model can be calculated as:
$\sum_{t=0}^{m} t(m-t) C_m^t p_j^t q_j^{m-t} = \sum_{t=1}^{m-1} t(m-t) C_m^t p_j^t q_j^{m-t} =$
$m(m-1) p_j q_j \sum_{t=0}^{m-2} C_{m-2}^t p_j^{t-1} q_j^{m-2-t} = m(m-1) p_j q_j$

The additional use of dispersions in this model brings more points. Combined with the Chebyshev inequality this gives intervals around the values s_j with a property that the random weights belong to these intervals with a strongly positive probability. This is our result for the point 1. The domain described by the above intervals is a rectangular area in the space of all weight vectors Ξ_m^n and the achieved property insists that there exist a proper random weight vector that belongs to the indicated rectangular area. Setting $s_1, s_2, \ldots, s_n$ arbitrarily, we receive corresponding rectangular area of different size and probability (it can be also empty). Unless attractive, in this form the property is not yet useful, because of we do not know if the rows of random matrix are different in this case.

Now consider the issue of 2. Consider a pair of random rows. The probability that particular j-th coordinates of this pair are identical is evidently $2p_j(1-p_j)$. The probability that the entire rows are identical equals $\prod_{j=1}^{n} 2p_j(1-p_j)$ and the corresponding probability that this rows are different will be $1-\prod_{j=1}^{n} 2p_j(1-p_j)$. For example, when $p_j = q_j = 1/2$ then we receive $\alpha = 1-\frac{1}{2^n}$. Let us consider m elements dropped onto the E^n. The probability that all pairs of these m elements will appear different is majored by $\beta = (1-\prod_{j=1}^{n} 2p_j(1-p_j))^{m^2}$ and $\gamma = (1-\frac{1}{2^n})^{m^2}$ is the value in particular case $p_j = 1/2$. m^2 is used as a raw estimate for number of pairs, C_m^2. For growing n, and relatively small m $\gamma \sim e^{-\frac{m^2}{2^n}}$. We suppose that γ is non zero then derive from this that there must exist a matrix with different rows. In general case there is a deviation from the $p_j = 1/2$ which can be maintained inserting a deviation coefficient δ so that $\beta = \gamma \cdot \delta$. Then, it is to determine the proper value of m so that the probability β is non zero that derives to the existence of a row-different matrix. Combining 1. and 2. for given $s_1, s_2, \dots, s_n$ we receive conditions and a rectangular area of Ξ_m^n around $s_1, s_2, \dots, s_n$, that contains a random row-different matrix. This probabilistic result can be extended inserting into the consideration the dispersion values.

Considerations above intend to get an additional knowledge about the row-different matrices using the probabilistic theory of combinatorics [14-17]. The objective is reasonable because the pure combinatorial approach is not able at the moment to give an efficient description of the column weighted row-different matrices. The probabilistic method gives a knowledge on random subsets, which might be useful as a complementary knowledge about a different object or a situation.

Consider in this regard the Sperner families. Many of existence issues on this are resolved – the maximal Sperner family, almost all Sperner families, weighted Sperner family (by recursive apply of Kruskal-Katona theorem [20, 21]). To understand the relation between the random sets and the Sperner property consider a bipartite graph with left side including all m-subsets and with right side that consists of all pairs of comparable vertices of E^n. Compute the average number of comparable pairs of vertices:

$$\varepsilon = \frac{\sum_{i=0}^{n} C_n^i (2^i - 1) C_{2^n-2}^{m-2}}{C_{2^n}^m} = \frac{m(m-1)}{2^n(2^n-1)}(3^n - 2^n)$$

Let $m^2 = o(1.333\ldots^n)$ then ε is nearly zero that indicates that there exist a Sperner family of size m, or in more precise that the random subset of this size is a Sperner Family.

5. CONCLUSION

Characterization of n-cube subsets in terms of the variable frequencies is considered. Diverse constraints on subsets generate a large number of problems that are interrelated. The general descriptive domain under the constraints is intensively investigated but the basic objective that is the well known Berge hypothesis about the hypergraph degree sequences still remains open. Characterization in terms of monotone Boolean functions (MBF) [10-13] is likely the main positive result in the domain. Due to the large number of MBF in practical domain we need complementary results that help with descriptions in diverse conditions. Several complexity relations are formulated in the paper. The homogeneous case of the problems is considered in detail. It is attempted, firstly, to apply the well known approach of probabilistic combinatorics in gaining complementary properties concerning the constraint based existence, restructuring and characterization type problems.

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