On Interval Cyclic Colorings of Bipartite Graphs

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ABSTRACT
A proper edge-coloring of a graph G with colors $1,\dots,t$ is called an interval (interval cyclic) t-coloring if all colors are used, and the edges incident to each vertex $v \in V(G)$ are colored by $d_G(v)$ consecutive colors (modulo t), where $d_G(v)$ is the degree of a vertex v in G. A graph G is interval (interval cyclically) colorable if it has an interval (interval cyclic) t-coloring for some positive integer t. An (a,b)-biregular bipartite graph G is a bipartite graph G with the vertices in one part all having degree a and the vertices in the other part all having degree b. In 1995, Toft conjectured that all biregular bipartite graphs are interval colorable. This conjecture remains open even for $(4,3)$-biregular bipartite graphs. Recently, Casselgren and Toft suggested the following weaker version of the Toft's conjecture: all biregular bipartite graphs are interval cyclically colorable. They also proved this conjecture for all $(8,4)$-biregular bipartite graphs. In this paper we prove the last conjecture for all (a,b)-biregular bipartite graphs when $(a,b) \in \{(5,3),(6,4),(7,4),(8,6)\}$.

Keywords
Edge-coloring, interval coloring, interval cyclic coloring, bipartite graph, biregular bipartite graph

1. INTRODUCTION
All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of G by $\Delta(G)$ and the chromatic index of G by $\chi'(G)$. An (a,b)-biregular bipartite graph G is a bipartite graph G with the vertices in one part all having degree a and the vertices in the other part all having degree b. The terms and concepts that we do not define can be found in [3, 14, 23].

A proper edge-coloring of a graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. A proper edge-coloring of a graph G with colors $1,\dots,t$ is an interval t-coloring [1] if all colors are used, and the edges incident to each vertex $v \in V(G)$ are colored by $d_G(v)$ consecutive colors. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. The concept of interval edge-coloring of graphs was introduced by Asratian and Kamalian [1] in 1987. In [1], they proved that if $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. Asratian and Kamalian also proved [1, 2] that if a triangle-free graph G has an interval t-coloring, then $t \leq |V(G)| - 1$. In [8, 9], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved that the complete bipartite graph $K_{m,n}$ has an interval t-coloring if and only if $m+n-\gcd(m,n) \leq t \leq m+n-1$, where $\gcd(m,n)$ is the greatest common divisor of m and n. In 1995, Toft conjectured that all biregular bipartite graphs are interval colorable. It is known that this conjecture is true for all (a,b)-biregular bipartite graphs with $\min\{a,b\} = 2$ [5, 6, 10]. In 2014, Casselgren and Toft [4] have proved that the conjecture is also true for all $(6,3)$-biregular bipartite graphs. Nevertheless, the conjecture remains open even for $(4,3)$-biregular bipartite graphs [7]. In [17], Petrosyan investigated interval colorings of complete graphs and hypercubes. In particular, he proved that if $n \leq t \leq \frac{n(n+1)}{2}$, then the hypercube Q_n has an interval t-coloring. Later, in [18], it was shown that the hypercube Q_n has an interval t-coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$. In [21], Sevast'janov proved that it is an NP-complete problem to decide whether a bipartite graph has an interval coloring or not.

A proper edge-coloring of a graph G with colors $1,\dots,t$ is called an interval cyclic t-coloring if all colors are used, and the edges incident to each vertex $v \in V(G)$ are colored by $d_G(v)$ consecutive colors modulo t. A graph G is interval cyclically colorable if it has an interval cyclic t-coloring for some positive integer t. For a graph $G \in \mathfrak{N}_c$, the least and the greatest values of t for which it has an interval cyclic t-coloring are denoted by $w_c(G)$ and $W_c(G)$, respectively. This type of edge-coloring under the name of "π-coloring" was first considered by Kotzig in [13], where he proved that every cubic graph has a π-coloring with 5 colors. However, the concept of interval cyclic edge-coloring of graphs was explicitly introduced by de Werra and Solot [22]. In [22], they proved that if G is an outerplanar bipartite graph, then G has an interval cyclic t-coloring for any $t \geq \Delta(G)$. In [15], Kubale and Nadolski showed that the problem of determining whether a given bipartite graph is interval cyclically colorable is NP-complete. Later, Nadolski

[16] showed that if G is interval colorable, then G has an interval cyclic $\Delta(G)$-coloring. He also proved that if G is a connected graph with $\Delta(G) = 3$, then $G \in \mathfrak{N}$ and $w_c(G) \leq 4$. In [11, 12], Kamalian investigated interval cyclic colorings of simple cycles and trees. For simple cycles and trees, he determined all possible values of t for which these graphs have an interval cyclic t-coloring. In [20], Petrosyan and Mkhitaryan investigated interval cyclic colorings of graphs. In particular, they proved that if G is a triangle-free graph with at least two vertices and $G \in \mathfrak{N}_c$, then $W_c(G) \leq |V(G)| + \Delta(G) - 2$, and this upper bound is sharp. They also showed that all complete bipartite and tripartite graphs are interval cyclically colorable.

In this paper, we first show that all bipartite graphs with maximum degree at most 4 are interval cyclically colorable. Next, we prove that: 1) if G is a $(5,3)$-biregular bipartite graph, then it has an interval cyclic coloring with at most 6 colors; 2) if G is a $(6,4)$-biregular bipartite graph, then G has an interval cyclic 6-coloring. Finally, we show that: 1) if G is a $(7,4)$-biregular bipartite graph, then it has an interval cyclic coloring with at most 8 colors; 2) if G is an $(8,6)$-biregular bipartite graph, then G has an interval cyclic 8-coloring.

2. MAIN RESULTS

First we consider bipartite graphs with a small maximum degree. It is known [5] that if G is a bipartite graph with $\Delta(G) \leq 3$, then $G \in \mathfrak{N}$. On the other hand, in [19] Petrosyan and Khachatrian proved that for any integer $\Delta \geq 11$, there exists a bipartite graph G such that $G \notin \mathfrak{N}$ and $\Delta(G) = \Delta$. The cases of Δ, $4 \leq \Delta \leq 10$, are still unsolved [7, 19]. Nevertheless, we show that all bipartite graphs with maximum degree 4 are interval cyclically colorable.

Theorem 1. *If G is a bipartite graph with $\Delta(G) = 4$, then $G \in \mathfrak{N}_c$ and $w_c(G) = 4$.*

In [4], Casselgren and Toft proved that all $(6,3)$-biregular bipartite graphs are interval colorable. Now we consider biregular bipartite graphs with maximum degree 5 or 6.

Theorem 2. *If G is a $(5,3)$-biregular bipartite graph, then $G \in \mathfrak{N}_c$ and $w_c(G) \leq 6$.*

Theorem 3. *If G is a $(6,4)$-biregular bipartite graph, then $G \in \mathfrak{N}_c$ and $w_c(G) = 6$.*

In [4], the authors proved that all $(8,4)$-biregular bipartite graphs are interval cyclically colorable. For biregular bipartite graphs with maximum degree 7 or 8, we prove that the following results hold.

Theorem 4. *If G is a $(7,4)$-biregular bipartite graph, then $G \in \mathfrak{N}_c$ and $w_c(G) \leq 8$.*

Theorem 5. *If G is an $(8,6)$-biregular bipartite graph, then $G \in \mathfrak{N}_c$ and $w_c(G) = 8$.*

3. ACKNOWLEDGEMENT

The work of the third author was made possible by a research grant from the Armenian National Science and Education Fund (ANSEF) based in New York, USA.

REFERENCES

[1] A.S. Asratian, R.R. Kamalian, "Interval colorings of edges of a multigraph", *Appl. Math. 5*, pp. 25-34, 1987 (in Russian).

[2] A.S. Asratian, R.R. Kamalian, "Investigation on interval edge-colorings of graphs", *J. Combin. Theory Ser. B 62*, pp. 34-43, 1994.

[3] A.S. Asratian, T.M.J. Denley, R. Haggkvist, "Bipartite Graphs and their Applications", *Cambridge University Press*, Cambridge, 1998.

[4] C.J. Casselgren, B. Toft, "On interval edge colorings of biregular bipartite graphs with small vertex degrees", *J. Graph Theory*, 2014, http://onlinelibrary.wiley.com/doi/10.1002/jgt.21841/

[5] H.M. Hansen, "Scheduling with minimum waiting periods", *MSc Thesis, Odense University*, Odense, Denmark, 1992 (in Danish).

[6] D. Hanson, C.O.M. Loten, B. Toft, "On interval colorings of bi-regular bipartite graphs", *Ars Combin. 50*, pp. 23-32, 1998.

[7] T.R. Jensen, B. Toft, "Graph Coloring Problems", *Wiley Interscience Series in Discrete Mathematics and Optimization*, 1995.

[8] R.R. Kamalian, "Interval colorings of complete bipartite graphs and trees", preprint, *Comp. Cen. of Acad. Sci. of Armenian SSR*, Yerevan, 1989 (in Russian).

[9] R.R. Kamalian, "Interval edge colorings of graphs", *Doctoral Thesis*, Novosibirsk, 1990.

[10] R.R. Kamalian, A.N. Mirumian, "Interval edge-colorings of bipartite graphs of some class", *Dokl. NAN RA 97*, pp. 3-5, 1997 (in Russian).

[11] R.R. Kamalian, "On cyclically-interval edge colorings of trees", *Buletinul of Academy of Sciences of the Republic of Moldova, Matematica 1(68)*, pp. 50-58, 2012.

[12] R.R. Kamalian, "On a number of colors in cyclically interval edge colorings of simple cycles", *Open J. Discrete Math. 3*, pp. 43-48, 2013.

[13] A. Kotzig, "1-Factorizations of cartesian products of regular graphs", *J. Graph Theory 3*, pp. 23-34, 1979.

[14] M. Kubale, "Graph Colorings", *American Mathematical Society*, 2004.

[15] M. Kubale, A. Nadolski, "Chromatic scheduling in a cyclic open shop", *European J. Oper. Res. 164* pp. 585-591, 2005.

[16] A. Nadolski, "Compact cyclic edge-colorings of graphs", *Discrete Math. 308*, pp. 2407-2417, 2008.

[17] P.A. Petrosyan, "Interval edge-colorings of complete graphs and n-dimensional cubes", *Discrete Math. 310*, pp. 1580-1587, 2010.

[18] P.A. Petrosyan, H.H. Khachatrian, H.G. Tananyan, “Interval edge-colorings of Cartesian products of graphs I”, *Discuss. Math. Graph Theory 33(3)*, pp. 613-632, 2013.

[19] P.A. Petrosyan, H.H. Khachatrian, “Interval non-edge-colorable bipartite graphs and multigraphs”, *J. Graph Theory 76*, pp. 200-216, 2014.

[20] P.A. Petrosyan, S.T. Mkhitaryan, “Interval cyclic edge-colorings of graphs”, *Preprint, Available at:* http://arxiv.org/abs/1411.0290

[21] S.V. Sevast'janov, “Interval colorability of the edges of a bipartite graph”, *Metody Diskret. Analiza 50*, pp. 61-72, 1990 (in Russian).

[22] D. de Werra, Ph. Solot, “Compact cylindrical chromatic scheduling”, *SIAM J. Disc. Math, Vol. 4, N4*, pp. 528-534, 1991.

[23] D.B. West, “Introduction to Graph Theory”, *Prentice-Hall*, New Jersey, 2001.