On Hamiltonian Bypasses in Digraphs with the Condition of Y. Manoussakis

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ABSTRACT

Let D be a strongly connected directed graph of order $n \geq 4$ which satisfies the following condition for every triple x, y, z of vertices such that x and y are non-adjacent: If there is no arc from x to z, then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n-2$. If there is no arc from z to x, then $d(x) + d(y) + d^-(x) + d^+(z) \geq 3n-2$. In [15] (J. of Graph Theory, **16**(5), 51-59, 1992) Y. Manoussakis proved that D is Hamiltonian. In [9] it was shown that D contains a pre-Hamiltonian cycle (i.e., a cycle of length $n-1$) or n is even and D is isomorphic to the complete bipartite digraph with partite sets of cardinalities of $n/2$ and $n/2$. In this paper we show that D contains also a Hamiltonian bypass (i.e., a subdigraph is obtained from a Hamiltonian cycle by reversing exactly one arc) or D is isomorphic to one tournament of order five.

Keywords

Digraphs, cycles, Hamiltonian cycles, Hamiltonian bypasses

1. INTRODUCTION

Terminology and notations not described below follows from [1]. A directed graph (digraph) D is Hamiltonian if it contains a Hamiltonian cycle, i.e., a cycle that includes every vertex of D. A Hamiltonian bypass in D is a subdigraph obtained from a Hamiltonian cycle by reversing exactly one arc. We now recall the following well-known degree conditions (Theorems 1.1-1.6) that guarantee that a digraph is Hamiltonian.

Theorem 1.1 (Nash-Williams [17]). *Let D be a digraph of order n such that for every vertex x, $d^+(x) \geq n/2$ and $d^-(x) \geq n/2$, then D is Hamiltonian.*

Theorem 1.2 (Ghouila-Houri [14]). *Let D be a strongly connected digraph of order n. If $d(x) \geq n$ for all vertices $x \in V(D)$, then D is Hamiltonian.*

Theorem 1.3 (Woodall [19]). *Let D be a digraph of order $n \geq 2$. If $d^+(x) + d^-(y) \geq n$ for all pairs of vertices x and y such that there is no arc from x to y, then D is Hamiltonian.*

Theorem 1.4 (Meyniel [16]). *Let D be a strongly connected digraph of order $n \geq 2$. If $d(x) + d(y) \geq 2n-1$ for all pairs of non-adjacent vertices in D, then D is Hamiltonian.*

It is easy to see that Meyniel's theorem is a common generalization of Nash-Williams', Ghouila-Houri's and Woodall's theorems. For a short proof of Theorem 1.4, see [5].

C. Thomassen [18] (for $n = 2k+1$) and S. Darbinyan [6] (for $n = 2k$) proved the following:

Theorem 1.5 (Thomassen [18], Darbinyan [6]). *If D is a digraph of order $n \geq 5$ with minimum degree at least $n-1$ and with minimum semi-degree at least $n/2-1$, then D is Hamiltonian (unless some extremal cases which are characterized).*

In view of the next theorems we need the following definitions.

Definition 1.1 [15]. *Let k be an integer. A digraph D of order $n \geq 3$ satisfies condition A_k if and only if for every triple of vertices x, y, z such that x and y are non-adjacent: If there is no arc from x to z, then $d(x)+d(y)+d^+(x)+d^-(z) \geq 3n-2+k$. If there is no arc from z to x, then $d(x)+d(y)+d^-(x)+d^+(z) \geq 3n-2+k$.*

Definition 1.2. *Let D_0 denote any digraph of order $n \geq 5$, n odd, such that $V(D_0) = A \cup B$, where $A \cap B = \emptyset$, A is an independent set with $(n+1)/2$ vertices, B is a set of $(n-1)/2$ vertices inducing any arbitrary subdigraph, and $e(A, B) = (n+1)(n-1)/2$.*

It is not difficult to check that D_0 satisfies condition A_{-1}, but has no Hamiltonian bypass.

Definition 1.3. *For any $k \in [1, n-2]$ let D_1 denotes a digraph of order $n \geq 4$, obtained from K^*_{n-k} and K^*_{k+1} by identifying a vertex of the first with a vertex of the second.*

It is not difficult to check that D_1 satisfies condition A_{-1}, but has no Hamiltonian bypass.

Definition 1.4. *By $T(5)$ we denote a tournament of order 5 with vertex set $V(T(5)) = \{x_1, x_2, x_3, x_4, y\}$ and arc set $A(T(5)) = \{x_i x_{i+1} / i \in [1,3]\}$ $\cup \{x_4 x_1, x_1 y, x_3 y, y x_2, y x_4, x_1 x_3, x_2 x_4\}$.*

$T(5)$ satisfies condition A_0, but has no Hamiltonian bypass.

Theorem 1.6 (Manoussakis [15]). *If a strongly connected digraph D of order $n \geq 4$ satisfies the condition A_0, then D is Hamiltonian.*
Benhocine [4] proved that if a digraph D satisfies the

condition of Nash-Williams' or Ghouila-Houri's or Woodall's theorem, then D contains a Hamiltonian bypass. In [4] also the following theorem was shown:

Theorem 1.7 (Benhocine [4]). *Every strongly 2-connected digraph of order n and with minimum degree at least $n-1$ contains a Hamiltonian bypass, unless D is isomorphic to a digraph of type D_0.*

In [7] the following theorem was proved:

Theorem 1.8 (Darbinyan [7]). *Let D be a strongly connected digraph of order $n \geq 3$. If $d(x) + d(y) \geq 2n-2$ for all pairs of non-adjacent vertices in D, then D contains a Hamiltonian bypass unless it is isomorphic to a digraph of the set $D_0 \cup \{D_1, T_5, C_3\}$, where C_3 is a directed cycle of length 3.*

For $n \geq 3$ and $k \in [2, n]$ by $D(n, k)$ we denote a digraph of order n obtained from a directed cycle C of length n by reversing exactly $k-1$ consecutive arcs. In [7] and [8] Darbinyan studied the problem of the existence of $D(n, 3)$ in digraphs with condition of Meyniel's theorem and in oriented graphs with the large in-degrees and out-degrees.

Theorem 1.9 (Darbinyan [7]). *Let D be a strongly connected digraph of order $n \geq 4$. If $d(x) + d(y) \geq 2n-1$ for all pairs of non-adjacent vertices in D, then D contains a $D(n, 3)$.*

Theorem 1.10 (Darbinyan [8]). *Let D be a oriented graph of order $n \geq 10$. If the minimum in-degree and out-degree of D at least $(n-3)/2$, then D contains a $D(n, 3)$.*

In [9] the following theorem was proved:

Theorem 1.11 (Darbinyan, Karapetyan). *Any strongly connected digraph D of order $n \geq 4$ satisfying condition A_0 contains a pre-Hamiltonian cycle (i.e., a cycle of length $n-1$) or n is even and D is isomorphic to the complete bipartite digraph with partite sets of cardinalities $n/2$ and $n/2$.*

In this paper using Theorem 1.11 we prove the following:

Theorem. *Any strongly connected digraph D of order $n \geq 4$ satisfying condition A_0 contains a Hamiltonian bypass unless D is isomorphic to the tournament $T(5)$.*

The following two examples show the sharpness of the condition of the theorem. The digraph consisting of the disjoint union of two complete digraphs with one common vertex shows that the bound in the above theorem is best possible and the digraph was obtained from a complete bipartite digraph after deleting one arc.

2. TERMINOLOGY AND NOTATIONS

In this paper we consider finite digraphs without loops and multiple arcs. For a digraph D, we denote by $V(D)$ the vertex set of D and by $A(D)$ the set of arcs in D. The arc of a digraph D directed from x to y is denoted by xy. For disjoint subsets A and B of $V(D)$ we define $A(A \to B)$ as the set $\{xy \in A(D)/x \in A, y \in B\}$. The out-neighborhood of a vertex x is the set $N^+(x) = \{y \in V(D)/xy \in A(D)\}$ and $N^-(x) = \{y \in V(D)/yx \in A(D)\}$ is the in-neighborhood of x. Similarly, if $A \subseteq V(D)$, then $N^+(x, A) = \{y \in A/xy \in A(D)\}$ and $N^-(x, A) = \{y \in A/yx \in A(D)\}$. The out-degree of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the in-degree of x. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The degree of the vertex x in D defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$).

The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). A cycle that contains all the vertices of D except one is a pre-Hamiltonian cycle. If P is a path containing a subpath from x to y we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y. For integers a and b, $a \leq b$, let $[a, b]$ denote the set of all integers which are not less than a and are not greater than b.

3. PRELIMINARIES

The following well-known simple Lemmas 3.1-3.4 are the basis of our results and other theorems on directed cycles and paths in digraphs. They will be used extensively in the proof of our result.

Lemma 3.1 [13]. *Let D be a digraph of order $n \geq 3$ containing a cycle C_m, $m \in [2, n-1]$. Let x be a vertex not contained in this cycle. If $d(x, C_m) \geq m+1$, then D contains a cycle C_k for all $k \in [2, m+1]$.*

The following lemma is a slight modification of a lemma by Bondy and Thomassen [5].

Lemma 3.2. *Let D be a digraph of order $n \geq 3$ containing a path $P := x_1 x_2 \ldots x_m$, $m \in [2, n-1]$ and let x be a vertex not contained in this path. If the following condition holds: $d(x, P) \geq m + |A(x \to x_1)| + |A(x_m \to x)|$, then there is an $i \in [1, m-1]$ such that $x_i x, x x_{i+1} \in D$ (the arc $x_i x_{i+1}$ is a partner of x), i.e., D contains a path $x_1 x_2 \ldots x_i x x_{i+1} \ldots x_m$ of length m (we say that x can be inserted into P or the path $x_1 x_2 \ldots x_i x x_{i+1} \ldots x_m$ is an extended path obtained from P with x).*

If in Lemma 3.1 and Lemma 3.2 instead of the vertex x consider a path Q, then we get the following Lemmas 3.3 and 3.4, respectively.

Lemma 3.3. *Let $C_k := x_1 x_2 \ldots x_k x_1$, $k \geq 2$, be a non-Hamiltonian cycle in a digraph D. Moreover, assume that there exists a path $Q := y_1 y_2 \ldots y_r$, $r \geq 1$, in $D - C_k$. If $d^-(y_1, C_k) + d^+(y_r, C_k) \geq k+1$, then for all $m \in [r+1, k+r]$ the digraph D contains a cycle C_m of length m with vertex set $V(C_m) \subseteq V(C_k) \cup V(Q)$.*

Lemma 3.4. *Let $P := x_1 x_2 \ldots x_k$, $k \geq 2$, be a non-Hamiltonian path in a digraph D. Moreover, assume that there exists a path $Q := y_1 y_2 \ldots y_r$, $r \geq 1$, in $D - P$. If $d^-(y_1, P) + d^+(y_r, P) \geq k + |A(y_1 \to x_1)| + |A(x_k \to y_r)|$, then there is a x_i, $i \in [1, k-1]$, such that $x_i y_1, y_r x_{i+1} \in D$ and D contains a path from x_1 to x_k with vertex set $V(P) \cup V(Q)$.*

In the proof of the our theorem we also need the following lemma which is a simple extension of a lemma due to Y. Manoussakis [15].

Lemma 3.5. *Let D be a digraph of order $n \geq 3$ satisfying condition A_0. Assume that there are two distinct pairs x, y and x, z of non-adjacent vertices in D. If $d(x) + d(y) \leq 2n - a$ for some integer $a \geq 1$, then $d(x)+d(z) \geq 2n-2+a/2$. In particular, if $d(x)+d(y) \leq 2n - 2$, then $d(x) + d(z) \geq 2n - 1$.*

Definition 3.1 ([1], [2]). *Let $Q = y_1y_2 \dots y_s$ be a path in a digraph D (possibly, $s = 1$) and let $P = x_1x_2 \dots x_t$, $t \geq 2$, be a path in $D - V(Q)$. Q has a partner on P if there is an arc (the partner of Q) x_ix_{i+1} such that $x_iy_1, y_sx_{i+1} \in D$. In this case the path Q can be inserted into P to give a new (x_1, x_t)-path with vertex set $V(P) \cup V(Q)$. The path Q has a collection of partners on P if there are integers $i_1 = 1 < i_2 < \dots < i_m = s + 1$ such that, for every $k = 2, 3, \dots, m$ the subpath $Q[y_{i_{k-1}}, y_{i_k - 1}]$ has a partner on P.*

Lemma 3.6 (Multi-Insertion Lemma [1], [2]). *Let $Q = y_1y_2 \dots y_s$ be a path in a digraph D (possibly, $s = 1$) and let $P = x_1x_2 \dots x_t$, $t \geq 2$, be a path in $D - V(Q)$. If Q has a collection of partners on P, then there is an (x_1, x_t)-path with vertex set $V(P) \cup V(Q)$.*
The following lemma is obvious.

Lemma 3.7. *Let D be a digraph of order $n \geq 3$ and let $C := x_1x_2 \dots x_{n-1}x_1$ be an arbitrary cycle of length $n - 1$ in D. If a vertex y is not on C and D contains no Hamiltonian bypass, then*
(i) $d^+(y, \{x_i, x_{i+1}\}) \leq 1$ and $d^-(y, \{x_i, x_{i+1}\}) \leq 1$ for all $i \in [1, n-1]$;
(ii) $d^+(y) \leq (n-1)/2$, $d^-(y) \leq (n-1)/2$ and $d(y) \leq n - 1$;
(iii) if $x_ky, yx_{k+1} \in D$, then $x_{i+1}x_i \notin D$ for all $x_i \neq x_k$.
Let D be a digraph of order $n \geq 3$ and let C_{n-1} be a cycle of length $n - 1$ in D. If for the vertex $y \notin C_{n-1}$, $d(y) \geq n$, then we say that C_{n-1} is a good cycle. Notice that, by Lemma 3.7(ii), if a digraph D contains a good cycle, then D also contains a Hamiltonian bypass.

4. THE OUTLINE OF THE PROOF OF THE MAIN RESULT

In the proof of our result we will use the following definition:

Definition 4.1. *Let $P_0 := x_1x_2 \dots x_m$, $m \geq 2$, be an (x_1, x_m)-path in D and let the vertices $y_1, y_2, \dots y_k$ be in $V(D) - V(P_0)$. For $i \in [1, k]$ we denote by P_i an (x_1, x_m)-path in D with vertex set $V(P_{i-1}) \cup \{y_j\}$ (if it exists), i.e, P_i is extended path obtained from P_{i-1} with some vertex y_j, where $y_j \notin V(P_{i-1})$. If $e + 1$ is the maximum possible number of these paths $P_0, P_1, \dots, P_e$, $e \in [0, k]$, then we say that P_e an extended path is obtained from P_0 with vertices $y_1, y_2, \dots, y_k$ as much as possible. Notice that P_i is an (x_1, x_m)-path of length $m + i - 1$ for all $i \in [0, e]$.*

The outline of the proof. Let D be a strongly connected digraph of order $n \geq 4$ satisfying condition A_0. By Theorem 1.1 the digraph D contains a cycle of length $n - 1$ or n is even and D is isomorphic to the complete bipartite digraph with partite sets of cardinalities of $n/2$ and $n/2$. If D is a complete bipartite digraph then it is easy to see that D has a Hamiltonian bypass. In the sequel, we assume that D contains a cycle of length $n-1$. Let $C = x_1x_2 \dots x_{n-1}x_1$ be an arbitrary cycle of length $n - 1$ in D and let $y \notin C$. It is a simple matter to check that for $n = 4$ the theorem is true. Further, let $n \geq 5$. Note that from condition A_0 and Lemma 5 it immediately follows that $d(y) \geq 3$. Now suppose, to the contrary, that D contains no Hamiltonian bypass (by Lemma 3.7(ii) it is clear that D also contains no good cycle).

For the cycle C and the vertex y frist we prove the following Claims 1- 7 below. Here we prove only Claims 1 and 3.

Claim 1. *$d(y, \{x_i\}) \leq 1$ for all $i \in [1, n - 1]$.*

Proof of Claim 1. Assume that the claim is not true. Without loss of generality, assume that $d(y, \{x_{n-1}\}) = 2$, i.e., $x_{n-1}y, yx_{n-1} \in D$. By Lemma 3.7(i), y is not adjacent to x_1 and x_{n-2}. Since $d(y) \geq 3$, we can assume that for some integers $a \geq 1$ and $b \geq 1$ the following holds:

$$d(y, \{x_1, x_2, \dots, x_a\}) =$$

$$= d(y, \{x_{n-2}, x_{n-3}, \dots, x_{n-b-1}\}) = 0, \quad (1)$$

and

$$min\{d(y, \{x_{a+1}\}), d(y, \{x_{n-b-2}\})\} \geq 1 \quad (2)$$

($x_{n-b-2} = x_{a+1}$ is possible). Now from Lemma 3.7(i) and (1) it follows that

$$d(y) = d(y, \{x_{n-1}\}) + d(y, C[x_{a+1}, x_{n-b-2}]) \leq$$

$$\leq n - b - a + 1. \quad (3)$$

If there is an (x_{a+1}, x_{n-1})-path P (respectively, an (x_{n-1}, x_{n-b-2})-path Q) with vertex set $V(C)$, then, since (2) and $d(y, \{x_{n-1}\}) = 2$, it is easy to see that D contains a Hamiltonian bypass. So, we may assume that there is no (x_{a+1}, x_{n-1})-path and there is no (x_{n-1}, x_{n-b-2})-path with vertex set $V(C)$. We extend the path $P_0 := C[x_{a+1}, x_{n-1}]$ (respectively, $P_0 := C[x_{n-1}, x_{n-b-2}]$) with vertices $x_1, x_2, \dots, x_a$ (respectively, $x_{n-b-1}, x_{n-b}, \dots, x_{n-2}$) as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_1, x_2, \dots, x_a\}$, $d \in [1, a]$, (respectively, $u_1, u_2, \dots, u_l \in \{x_{n-b-1}, x_{n-b}, \dots, x_{n-2}\}$, $l \in [1, b]$) are not on the extended path P_e. Therefore using Lemma 3.2(i), we obtain that

$$d(z_i) \leq n + d - 2 \quad \text{and} \quad d(u_j) \leq n + l - 2 \quad (4)$$

for all $i \in [1, d]$ and $j \in [1, l]$. Since $d \leq a + b - 1$ and $l \leq a + b - 1$, from inequalities (3) and (4) it follows that
$d(y) + d(z_i) \leq 2n - 1 + d - a - b \leq 2n - 2$
and $d(y) + d(u_j) \leq 2n - 1 + l - a - b \leq 2n - 2$.
The last two inequalities contradicts Lemma 3.5 since y, z_i and y, u_j are two distinct pairs of nonadjacent vertices. Claim 1 is proved. □

Claim 2. *$d(y) \leq n - 2$.*

Claim 3. *Suppose that $d(y, C[x_{l+1}, x_{k-1}]) = 0$ and y is adjacent to x_l and x_k, where x_l and x_k are distinct and $a + 2 := |C[x_l, x_k]| \geq 3$. Then the following holds:*
(i) if $x_ly, x_ky \in D$ or $yx_l, yx_k \in D$, then there is a vertex $u \in C[x_{l+1}, x_{k-1}]$ such that $d(y) + d(u) \leq 2n - 3$;
(ii) if $x_ly, yx_k \in D$, then there is an (x_k, x_l)-path with vertex set $V(C) - \{u\}$, where u is some vertex of $C[x_{l+1}, x_{k-1}]$ and $d(u) \leq n - 1$. In particular, $d(y) + d(u) \leq 2n - 3$.
(iii) if $x_ly, yx_k \in D$ (or $yx_l, yx_k \in D$ or $x_ly, x_ky \in D$), then there are no x_i and x_j such that $C[x_i, x_j] \neq C[x_l, x_k]$, $b := |C[x_i, x_j]| \geq 3$, $d(y, C[x_{i+1}, x_{j-1}]) = 0$

and a) $x_iy, x_jy \in D$ or b) $yx_i, yx_j \in D$ or c) $x_iy, yx_j \in D$.

Proof of Claim 3. By Claim 1, $d(y) \leq n - a - 1$.

(i). It is not difficult to see that there is no (x_k, x_l)-path with vertex set $V(C)$ (for otherwise D would contain a Hamiltonian bypass). We extend the path $P_0 := C[x_k, x_l]$ with vertices $x_{l+1}, x_{l+2}, \dots, x_{k-1}$ as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_{l+1}, x_{l+2}, \dots, x_{k-1}\}$, $d \in [1, a]$, are not on the obtained extended path P_e. Hence, using Lemma 3.2(i), we obtain that $d(z_i) \leq n+d-2$ (let $u := z_1$). Therefore, for all $i \in [1, d]$

$$d(y) + d(z_i) \leq n - a - 1 + n + d - 2 \leq 2n - 3. \quad (5)$$

(ii). Assume, without loss of generality, that $x_{n-1}y, yx_{a+1} \in D$ (i.e., $x_l = x_{n-1}$ and $x_k = x_{a+1}$) and $d(y, C[x_1, x_a]) = 0$ where $a \in [1, n-4]$. If $a = 1$, then Claim 3(ii) is clearly true. So, we can assume that $a \geq 2$. We extend the path $P_0 := C[x_{a+1}, x_{n-1}]$ with vertices $x_1, x_2, \dots, x_a$ as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_1, x_2, \dots, x_a\}$ are not on the extended path P_e. We claim that $d = 0$ or $d = 1$. Indeed, if $d \geq 2$, then for the vertices z_1 and z_2 inequality (5) holds, which contradicts Lemma 3.5. Therefore $d = 0$ or $d = 1$. If $d = 1$, then $d(z_1) \leq n - 1$ (let $u := z_1$) and P_e is an (x_{a+1}, x_{n-1})-path with vertex set $V(C) - \{u\}$, and if $d = 0$, then $e \geq 2$, P_{e-1} is an (x_{a+1}, x_{n-1})-path with vertex set $V(C) - \{u\}$, where now u is some vertex of $C[x_1, x_a]$. Then the path P_{e-1} together with the arcs $x_{n-1}y$ and yx_{a+1} form a cycle of length $n - 1$. Therefore, $d(u) \leq n - 1$ since D contains no good cycle. It is clear that $d(y) + d(u) \leq 2n - 3$.
(iii). Assume that Claim 3(iii) is not true. From Claims 3(i) and 3(ii) it follows that there are two distinct vertices $u \in C[x_{l+1}, x_{k-1}]$ and $v \in C[x_{i+1}, x_{j-1}]$ such that $d(y) + d(u) \leq 2n - 3$ and $d(y) + d(v) \leq 2n - 3$. The last two inequalities contradict Lemma 3.5, since y, u and y, v are two distinct pairs of non-adjacent vertices. Claim 3 is proved. □

Claim 4. *There are no two distinct vertices x_i and x_j such that $x_iy, x_jy \in D$ (or $yx_i, yx_j \in D$), $|C[x_i, x_j]| \geq 3$ and $d(y, C[x_{i+1}, x_{j-1}]) = 0$.*

Claim 5. *Let $x_ry, yx_k \in D$ and $d(y, C[x_{r+1}, x_{k-1}]) = 0$ for some $r, k \in [1, n-1]$, where $3 \leq |C[x_r, x_k]| \leq n - 2$. Then the vertices y and x_{k+1} are non-adjacent.*

Claim 6. *If $x_ly \in D$ and $d(y, C[x_{l+1}, x_{l+a}]) = 0$, where $a \in [1, n-4]$, then $yx_{l+a+1} \notin D$.*
Claim 7. *If $yx_l \in D$ and $d(y, C[x_{l+1}, x_{l+a}]) = 0$ with $a \in [1, n-4]$, then $x_{l+a+1}y \notin D$.* □

Now using Claims 1-7 we will complete the proof of Theorem 12. From Claims 1 and 2 it follows that there are two distinct vertices x_k, x_l such that $|C[x_k, x_l]| \geq 3$, y is adjacent to x_k, x_l and $d(y, C[x_{k+1}, x_{l-1}]) = 0$. Therefore, one of the following cases holds: (i) $x_ky, x_ly \in D$; (ii) $yx_k, yx_l \in D$; (iii) $x_ky, yx_l \in D$; (iv) $yx_k, x_ly \in D$. On the other hand, if D has no Hamiltonian bypass, then Claims 4, 6 and 7 imply that each of these cases is impossible. Thus we have a contradiction which proves the theorem. □

5. CONCLUDING REMARKS

Each of Theorems 1.1-1.6 imposes a degree condition on all pairs of non-adjacent vertices (or on all vertices). In the following three theorems a degree condition is imposed only for some pairs of non-adjacent vertices. In each of the conditions (Theorems 5.1-5.4) below D is a strongly connected digraph of order n.

Theorem 5.1 (Bang-Jensen, Gutin, H.Li [2]). *Suppose that $min\{d(x), d(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for any pair of non-adjacent vertices x, y with a common in-neighbour, then D is Hamiltonian.*

Theorem 5.2 (Bang-Jensen, Gutin, H.Li [2]). *Suppose that $min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n$ for any pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.*

Theorem 5.3 (Bang-Jensen, Guo, Yeo [3]). *Suppose that $d(x)+d(y) \geq 2n-1$ and $min\{d^+(x)+d^-(y), d^-(x)+d^+(y)\} \geq n - 1$ for any pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.*
Note that Theorem 5.3 generalizes Theorem 5.2.
In [10] the following results were proved:
(i) *if the minimum semi-degree of D at least two and D satisfies the condition of Theorem 5.1* or
(ii) *D is not a directed cycle and satisfies the condition of Theorem 5.2, then either D contains a pre-Hamiltonian cycle or n is even and D is isomorphic to the complete bipartite digraph with partite sets of cardinalities $n/2$ and $n/2$ or to the complete bipartite digraph minus one arc with partite sets of cardinalities $n/2$ and $n/2$.*
In [11] it was proved that
if D is not a directed cycle and satisfies the condition of Theorem 5.34, then D contains a pre-Hamiltonian cycle or a cycle of length $n - 2$.
We pose the following problem:

Problem. *Characterize those digraphs which satisfy the condition of Theorem 5.1 (or 5.2 or 5.3) but has no Hamiltonian bypass.*

In [12] the following theorem was proved:

Theorem 5.4. (Darbinyan, Karapetyan) *Suppose that $min\{d(x), d(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for any pair of non-adjacent vertices x, y with a common in-neighbour. If $n \geq 6$ and the minimum out-degree of D at least two and the minimum in-degree of D at least three, then D contains a Hamiltonian bypass.*
We believe that Theorem 5.4 also is true if we require that minimum in-degree at least two instead of three.

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