On pre-Hamiltonian Cycles in Hamiltonian Digraphs

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ABSTRACT

Let D be a strongly connected directed graph of order $n \geq 4$. In [14] (J. of Graph Theory, Vol.16, No. 5, 51-59, 1992) Y. Manoussakis proved the following theorem: Suppose that D satisfies the following condition for every triple x, y, z of vertices such that x and y are nonadjacent: If there is no arc from x to z, then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 2$. If there is no arc from z to x, then $d(x) + d(y) + d^-(x) + d^+(z) \geq 3n - 2$. Then D is Hamiltonian. In this paper we show that: If D satisfies the condition of Manoussakis' theorem, then D contains a pre-Hamiltonian cycle (i.e., a cycle of length $n - 1$) or n is even and D is isomorphic to the complete bipartite digraph with partite sets of cardinalities $n/2$ and $n/2$.

Keywords

Digraphs, cycles, Hamiltonian cycles, pre-Hamiltonian cycles, longest non-Hamiltonian cycles

1. INTRODUCTION

A directed graph (digraph) D is Hamiltonian if it contains a Hamiltonian cycle, i.e., a cycle of length n, and is pancyclic if it contains cycles of all lengths m, $3 \leq m \leq n$, where n is the number of vertices in D. We recall the following well-known degree conditions (Theorems 1-8) that guarantee that a digraph is Hamiltonian. In each of the conditions (Theorems 1-8) below D is a strongly connected digraph of order n:

Theorem 1. *(Ghouila-Houri [12]). If $d(x) \geq n$ for all vertices $x \in V(D)$, then D is Hamiltonian.*

Theorem 2. *(Woodall [18]). If $d^+(x) + d^-(y) \geq n$ for all pairs of vertices x and y such that there is no arc from x to y, then D is Hamiltonian.*

Theorem 3. *(Meyniel [15]). If $n \geq 2$ and $d(x) + d(y) \geq 2n - 1$ for all pairs of nonadjacent vertices in D, then D is Hamiltonian.*

It is easy to see that Meyniel's theorem is a common generalization of Ghouila-Houri's and Woodall's theorems. For a short proof of Theorem 3, see [5].

C. Thomassen [17] (for $n = 2k + 1$) and S. Darbinyan [7] (for $n = 2k$) proved the following theorem below.

Before stating it (for any integer $m \geq 2$) we need to introduce some additional notations.

$H(m, m)$ denotes the set of digraphs D of order $2m$ with vertex set $A \cup B$ such that $\langle A \rangle \equiv \langle B \rangle \equiv K^*_m$, there is no arc from B to A, $d^+(x, B) \geq 1$ and $d^-(y, A) \geq 1$ for every vertices $x \in A$ and $y \in B$.

$H(m, m-1, 1)$ denotes the set of digraphs D of order $2m$ with vertex set $A \cup B \cup \{a\}$ such that $|A| = |B| + 1 = m$, $\langle B \cup \{a\} \rangle \subseteq K^*_m$ (i.e., $\langle B \cup \{a\} \rangle$ is an arbitrary digraph) the subdigraph $\langle A \rangle$ has no arc, D contains all possible arcs between A and B and either $N^-(a) = B$ and $A \subseteq N^+(a)$, or $N^+(a) = B$ and $A \subseteq N^-(a)$.

$H(2m)$ denotes a digraph of order $2m$ with vertex set $A \cup B \cup \{x, y\}$ such that $\langle A \cup \{x\} \rangle \equiv \langle B \cup \{y\} \rangle \equiv K^*_m$, there is no arc between A and B, $H(2m)$ also contains all arcs of the form ya, bx, where $a \in A$ and $b \in B$, and either the arc xy or both arcs xy and yx.

Let D_6 be a digraph with vertex set $\{x_1, x_2, \ldots, x_5, x\}$ and arc set
$\{x_i x_{i+1} \,/ 1 \leq i \leq 4\} \cup \{x x_i / \, 1 \leq i \leq 3\}$
$\cup \{x_1 x_5, x_2 x_5, x_5 x_1, x_5 x_4, x_3 x_2, x_3 x, x_4 x_1, x_4 x\}$.

By D'_6 we denote a digraph obtained from D_6 by adding the arc $x_2 x_4$. Note that the digraphs D_6 and D'_6 both are not Hamiltonian and each of D_6 and D'_6 contains a cycle of length 5.

Theorem 4. *(Thomassen [17], Darbinyan [7]). If D is a digraph of order $n \geq 5$ with minimum degree at least $n - 1$ and with minimum semi-degree at least $n/2 - 1$. Then D is Hamiltonian unless*
(i) D is isomorphic to D_5 or D_7 or $[(K_m \cup K_m) + K_1]^*$ or $K^*_{m,m+1} \subseteq D \subseteq [K_m + \overline{K}_{m+1}]^*$, if $n = 2m + 1$;
(ii) $D \in H(m, m) \cup H(m, m-1, 1) \cup$
$\cup \{H(2m), H'(2m), D_6, D'_6, \overleftarrow{D_6}, \overleftarrow{D'_6}\}$, if $n = 2m$. (The digraphs D_5 and D_7 are well known and for their definitions, see, for example, [17]).

Definition 1. [14]. *Let k be an arbitrary nonnegative integer. A digraph D satisfies the condition A_k if and only if for every triple x, y, z of vertices such that x and y are nonadjacent: If there is no arc from x to z, then $d(x)+d(y)+d^+(x)+d^-(z) \geq 3n-2+k$. If there is no arc from z to x, then $d(x)+d(y)+d^-(x)+d^+(z) \geq 3n-2+k$.*

Theorem 5. *(Y. Manoussakis [14]). If a digraph D of order $n \geq 4$ satisfies the condition A_0, then D is Hamiltonian.*

Each of these theorems imposes a degree condition on all pairs of nonadjacent vertices (or on all vertices). In the following three theorems imposes a degree condition only for some pairs of nonadjacent vertices.

Theorem 6. *(Bang-Jensen, Gutin, H.Li [2]). Suppose that $min\{d(x), d(y)\} \geq n-1$ and $d(x)+d(y) \geq 2n-1$ for any pair of nonadjacent vertices x, y with a common in-neighbour, then D is Hamiltonian.*

Theorem 7. *(Bang-Jensen, Gutin, H.Li [2]). Suppose that $min\{d^+(x)+d^-(y), d^-(x)+d^+(y)\} \geq n$ for any pair of nonadjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.*

Theorem 8. *(Bang-Jensen, Guo, Yeo [3]). Suppose that $d(x)+d(y) \geq 2n-1$ and $min\{d^+(x)+d^-(y), d^-(x)+d^+(y)\} \geq n-1$ for any pair of nonadjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.*

Note that Theorem 8 generalizes Theorem 7.

In [11, 16, 6, 8] it was shown that if a digraph D satisfies the condition one of Theorems 1, 2, 3 and 4, respectively, then D also is pancyclic (unless some extremal cases which are characterized). It is natural to set the following problem:
Characterize those digraphs which satisfy the conditions of Theorem 6 (7, 8) but are not pancyclic.
In many papers (as well as, in the mentioned papers), the existence of a pre-Hamiltonian cycle (i.e., a cycle of length $n-1$) is essential to show whether a given digraph (graph) is pancyclic or not. This indicates that the existence of a pre-Hamiltonian cycle in the a digraph (graph) makes the pancyclic problem significantly easer, in a sense. For the digraphs which satisfy the conditions of Theorem 6 or 7 or 8 in [9] and [10] the following results are proved:
(i) if the minimum semi-degree of a digraph D at least two and D satisfies the conditions of Theorem 6 or a digraph D is not directed cycle and satisfies the conditions of Theorem 7, then either D contains a pre-Hamiltonian cycle (i.e., a cycle of length $n-1$) or n is even and D is isomorphic to the complete bipartite digraph $K^*_{n/2,n/2}$ or to the complete bipartite digraph $K^*_{n/2,n/2}$ minus one arc.
(ii) if a digraph D is not a directed cycle and satisfies the conditions of Theorem 8, then D contains a pre-Hamiltonian cycle or a cycle of length $n-2$.
In [14] the following conjecture was proposed:

Conjecture . Any strongly connected digraph satisfying the condition A_3 is pancyclic.

In this paper using some claims of the proof of Theorem 5 (see [14]) we prove the following:

Theorem 9. *Any strongly connected digraph D on $n \geq 4$ vertices satisfying the condition A_0 contains a pre-Hamiltonian cycle or n is even and D is isomorphic to the complete bipartite digraph $K^*_{n/2,n/2}$.*

The following examples show the sharpness of the bound $3n-2$ in the theorem. The digraph consisting of the disjoint union of two complete digraphs with one common vertex or the digraph obtained from a complete bipartite digraph after deleting one arc show that the bound $3n-2$ in the above theorem is best possible.

2. TERMINOLOGY AND NOTATIONS

We shall assume that the reader is familiar with the standard terminology on the directed graphs (digraph) and refer the reader to [1] for terminology not discussed here. In this paper we consider finite digraphs without loops and multiple arcs. For a digraph D, we denote by $V(D)$ the vertex set of D and by $A(D)$ the set of arcs in D. The order of D is the number of its vertices. Often we will write D instead of $A(D)$ and $V(D)$. The arc of a digraph D directed from x to y is denoted by xy. For disjoint subsets A and B of $V(D)$ we define $A(A \to B)$ as the set $\{xy \in A(D)/x \in A, y \in B\}$ and $A(A, B) = A(A \to B) \cup A(B \to A)$. If $x \in V(D)$ and $A = \{x\}$ we write x instead of $\{x\}$. If A and B are two disjoint subsets of $V(D)$ such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A \to B$. The out-neighborhood of a vertex x is the set $N^+(x) = \{y \in V(D)/xy \in A(D)\}$ and $N^-(x) = \{y \in V(D)/yx \in A(D)\}$ is the in-neighborhood of x. Similarly, if $A \subseteq V(D)$, then $N^+(x, A) = \{y \in A/xy \in A(D)\}$ and $N^-(x, A) = \{y \in A/yx \in A(D)\}$. The out-degree of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the in-degree of x. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The degree of the vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$). The subdigraph of D induced by a subset A of $V(D)$ is denoted by $\langle A \rangle$. The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs x_ix_{i+1}, $i \in [1, m-1]$ (respectively, x_ix_{i+1}, $i \in [1, m-1]$, and x_mx_1), is denoted by $x_1x_2 \cdots x_m$ (respectively, $x_1x_2 \cdots x_mx_1$). We say that $x_1x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)-path. For a cycle $C_k := x_1x_2 \cdots x_kx_1$ of length k, the subscripts considered modulo k, i.e., $x_i = x_s$ for every s and i such that $i \equiv s \pmod k$. A cycle that contains all the vertices of D (respectively, all the vertices of D except one) is a Hamiltonian cycle (respectively, is a pre-Hamiltonian cycle). The concept of the pre-Hamiltonian cycle was given in [13]. If P is a path containing a subpath from x to y we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y. A digraph D is strongly connected (or, just, strong) if there exists a path from x to y and a path from y to x for every pair of distinct vertices x, y. For an undirected graph G, we denote by G^* the symmetric digraph obtained from G by replacing every edge xy with the pair xy, yx of arcs. $K_{p,q}$ denotes the complete bipartite graph with partite sets of cardinalities p and q. Two distinct vertices x and y are adjacent if $xy \in A(D)$ or $yx \in A(D)$ (or both). For integers a and b, $a \leq b$, let $[a, b]$ denote the set of all integers which are not less than a and are not greater than b. Let C be a non-Hamiltonian cycle in digraph D. An (x, y)-path P is a C-bypass if $|V(P)| \geq 3$, $x \neq y$ and $V(P) \cap V(C) = \{x, y\}$.

3. PRELIMINARIES

The following well-known simple Lemmas 3.1-3.4 are the basis of our results and other theorems on directed cycles and paths in digraphs. They will be used extensively in the proofs of our results.
Lemma 1 [11]. *Let D be a digraph of order $n \geq 3$ containing a cycle C_m, $m \in [2, n-1]$. Let x be a vertex not contained in this cycle. If $d(x, C_m) \geq m+1$, then D contains a cycle C_k for all $k \in [2, m+1]$.*
The following lemma is a slight modification of a lemma by Bondy and Tomassen [5].
Lemma 2. Let D be a digraph of order $n \geq 3$ containing

a path $P := x_1x_2 \dots x_m$, $m \in [2, n-1]$ and let x be a vertex not contained in this path. If one of the following conditions holds:
(i) $d(x, P) \geq m + 2$;
(ii) $d(x, P) \geq m + 1$ and $xx_1 \notin D$ or $x_mx \notin D$;
(iii) $d(x, P) \geq m$, $xx_1 \notin D$ and $x_mx \notin D$,
then there is an $i \in [1, m-1]$ such that $x_ix, xx_{i+1} \in D$, i.e., D contains a path $x_1x_2 \dots x_ixx_{i+1} \dots x_m$ of length m (we say that x can be inserted into P or the path $x_1x_2 \dots x_ixx_{i+1} \dots x_m$ is an extended path from P with x).

If in Lemma 1 and Lemma 2 instead of the vertex x consider a path Q, then we get the following Lemmas 3 and 3.4, respectively.

Lemma 3. Let $C_k := x_1x_2 \dots x_kx_1$, $k \geq 2$, be a non-Hamiltonian cycle in a digraph D. Moreover, assume that there exists a path $Q := y_1y_2 \dots y_r$, $r \geq 1$, in $D - C_k$. If $d^-(y_1, C_k) + d^+(y_r, C_k) \geq k + 1$, then for all $m \in [r+1, k+r]$ the digraph D contains a cycle C_m of length m with vertex set $V(C_m) \subseteq V(C_k) \cup V(Q)$.

Lemma 4. Let $P := x_1x_2 \dots x_k$, $k \geq 2$, be a non-Hamiltonian path in a digraph D. Moreover, assume that there exists a path $Q := y_1y_2 \dots y_r$, $r \geq 1$, in $D - P$. If $d^-(y_1, P) + d^+(y_r, P) \geq k + d^-(y_1, \{x_k\}) + d^+(y_r, \{x_1\})$, then D contains a path from x_1 to x_k with vertex set $V(P) \cup V(Q)$.

For the proof of our result we also need the following
Lemma 5 [14]. *Let D be a digraph on $n \geq 3$ vertices satisfying the condition A_0. Assume that there are two distinct pairs of nonadjacent vertices x, y and x, z in D. Then either $d(x) + d(y) \geq 2n-1$ or $d(x) + d(z) \geq 2n-1$.*

4. THE OUTLINE OF THE PROOF OF THEOREM 9

In the proof of Theorem 9 we often will use the following definition:

Definition 2. Let $P_0 := x_1x_2 \dots x_m$, $m \geq 2$, be an arbitrary (x_1, x_m)-path in a digraph D and let $y_1, y_2, \dots y_k \in V(D) - V(P_0)$. For $i \in [1, k]$ we denote by P_i an (x_1, x_m)-path in D with vertex set $V(P_{i-1}) \cup \{y_j\}$ (if it exists) such that P_i is an extended path obtained from P_{i-1} with some vertex y_j, where $y_j \notin V(P_{i-1})$. If $e+1$ is the maximum possible number of these paths $P_0, P_1, \dots, P_e$, $e \in [0, k]$, then we say that P_e is an extended path obtained from P_0 with vertices $y_1, y_2, \dots, y_k$ as much as possible. Notice that P_i ($i \in [0, e]$) is an (x_1, x_m)-path of length $m + i - 1$.

Proof of Theorem 9. Let $C := x_1x_2 \dots x_kx_1$ be a longest non-Hamiltonian cycle in D of length k, and let C be chosen so that $\langle V(D) - V(C) \rangle$ has the minimum number of connected components. Suppose that $k \leq n-2$ and $n \geq 5$ (the case $n = 4$ is trivial). It is easy to show that $k \geq 3$. We will prove that D is isomorphic to the complete bipartite digraph $K^*_{n/2,n/2}$. Put $R := V(D) - V(C)$. Let $R_1, R_2, \dots, R_q$ be the connected components of $\langle R \rangle$ (i.e., if $q \geq 2$, then for any pair i, j, $i \neq j$, there is no arc between R_i and R_j). In [14] it was proved that for any R_i, $i \in [1, q]$, the subdigraph $\langle V(C) \cup V(R_i) \rangle$ contains a C-bypass. (The existence of a C-bypass also follows from Bypass Lemma (see [4]), since $\langle V(C) \cup V(R_i) \rangle$ is strong and condition A_0 implies that the underlying graph of the subdigraph $\langle V(C) \cup V(R_i) \rangle$ is 2-connected). Let $P := x_my_1y_2 \dots y_{t_i}x_{m+\lambda_i}$ be a C-bypass in $\langle V(C) \cup V(R_i) \rangle$ ($i \in [1, q]$ is arbitrary) and λ_i is considered to be minimum in the sense that there is no C-bypass $x_au_1u_2 \dots u_{l_i}x_{a+r_i}$ in $\langle V(C) \cup V(R_i) \rangle$ such that $r_i < \lambda_i$ and $\{x_a, x_{a+r_i}\}$ is a subset of $\{x_m, x_{m+1}, \dots, x_{m+\lambda_i}\}$.
We will distinguish two cases, according as there is a λ_i, $i \in [1, q]$, such that $\lambda_i = 1$ or not.
Assume first that $\lambda_i \geq 2$ for all $i \in [1, q]$. For this case one can show that (the proofs are the same as the proofs of Case 1, Lemma 2.3 and Claim 1 in [14]) if $\lambda_i \geq 2$, then $t_i = |R_i| = 1$, in $\langle V(C) \rangle$ there is an $(x_{m+\lambda_i}, x_m)$-path (say, P') of length $k-2$ with vertex set $V(P') = V(C) - \{z_i\}$, where $z_i \in \{x_{m+1}, x_{m+2}, \dots, x_{m+\lambda_i - 1}\}$ and $d(y_1) + d(z_i) \leq 2n-2$ (note that y_1 and z_i are nonadjacent). From $|R| \geq 2$ and $|R_i| = 1$ (for all i) it follows that $q \geq 2$. If $u \in R_2$, then $d(u) = d(u, C) \leq k$ (by Lemma 1) and $d(z_1, R) = 0$ (by minimality of q), in particular, the vertices z_1 and u are nonadjacent. Therefore $d(z_1) = d(z_1, C) \leq k$ and $d(z_1) + d(u) \leq 2n-2$. This in connection with $d(y_1) + d(z_1) \leq 2n-2$ contradicts Lemma 5.
Assume second that $\lambda_i = 1$ for all $i \in [1, q]$. It is clear that $q = 1$. Put $t := t_1$ and $\lambda := \lambda_1 = 1$.
Observe that if $v_1v_2 \dots v_j$ (maybe, $j = 1$) is a path in $\langle R \rangle$ and $x_iv_1 \in D$, then $v_jx_{i+j} \notin D$ since C is a longest non-Hamiltonian cycle in D and $k \leq n-2$. We shall use this often, without mentioning this explicitly.
The following claim follows immediately from $\lambda = 1$ and the maximality of C.

Claim 1. *$R = \{y_1, y_2, \dots, y_t\}$ (i.e., $t = n - k \geq 2$), $y_1y_2 \dots y_t$ is a Hamiltonian path in $\langle R \rangle$ and if $1 \leq i < j - 1 \leq t - 1$, then $y_iy_j \notin D$.*
Frist we proved Claims 2-4.

Claim 2. *(i). If $x_iy_1 \in D$, then*
$d^-(x_{i+1}, \{y_1, y_2, \dots, y_{t-1}\}) = 0$;
(ii). If $y_tx_{i+1} \in D$, then
$d^+(x_i, \{y_2, y_3, \dots, y_t\}) =$
$= d^+(x_{i-1}, \{y_1, y_2, \dots, y_{t-1}\}) = 0$;
(iii). $d(y_1, C) \leq k$, $d(y_t, C) \leq k$ and $d(y_j, C) \leq k - 1$ for all $j \in [2, t-1]$ (by Lemma 3.2(iii) and Claim 2(ii) since $\lambda = 1$).

Claim 3. *Assume that $\langle R \rangle$ is strong. If $d^+(x_i, R) \geq 1$, $d^-(x_j, R) \geq 1$ and $|C[x_i, x_j]| \geq 3$ for some two distinct vertices x_i, x_j ($i, j \in [1, k]$), then the following holds:*
(i) $d^-(x_{j-1}, R) \neq 0$ or $A(R, C[x_{i+1}, x_{j-2}]) \neq \emptyset$;
(ii) $d^+(x_{i+1}, R) \neq 0$) or $A(R, C[x_{i+2}, x_{j-1}]) \neq \emptyset$.
(Here if $|C[x_i, x_j]| = 3$, then $C[x_{i+1}, x_{j-2}] = \emptyset$ and $C[x_{i+2}, x_{j-1}] = \emptyset$).
In particular, from Claim 3 it immediately follows the following

Claim 4. *Assume that $\langle R \rangle$ is strong and $d^+(x_i, R) \geq 1$, $d^-(x_j, R) \geq 1$ for some two distinct vertices x_i and x_j. Then the following holds:*
(i) if $|C[x_i, x_j]| \geq 3$, then $A(R, C[x_{i+1}, x_{j-1}]) \neq \emptyset$;
(ii) if $|C[x_i, x_j]| = 3$, then $d^+(x_{i+1}, R) \geq 1$ and
$d^-(x_{j-1}, R) \geq 1$.
Now we divide the proof of the theorem into two parts: $k \leq n-3$ and $k = n-2$.

Part 1. $k \leq n-3$, i.e., $t \geq 3$.
For this part we will proved the following Claims 5-10 below.

Claim 5. *Let $t \geq 3$ and $y_ty_1 \in D$. Then the following*

holds
(i) if x_iy_1D, then $d^-(x_{i+2}, R) = 0$; (ii) if $y_tx_i \in D$, then $d^+(x_{i-2}, R) = 0$, where $i \in [1, k]$.

Claim 6. *If $t \geq 3$ and the vertices y_1, y_t are nonadjacent, then $t = 3$ and y_3y_2, $y_2y_1 \in D$.*

Claim 7. *If $t \geq 3$, then $y_ty_1 \in D$.*

Claim 8. *If $t \geq 3$ and for some $i \in [1, k]$ x_iy_1, then $A(R \to C[x_{i+2}, x_{i-1}]) = \emptyset$.*

Claim 9. *If $t \geq 3$, x_1y_1 and $y_tx_2 \in D$, then $d^-(x_1, R) = 0$.*

Claim 10. *If $t \geq 3$, x_1y_1 and $y_tx_2 \in D$, then $A(\{x_3, x_4, \dots, x_k\} \to R) = \emptyset$.*
Using Claims 7-10 we can complete the proof of Theorem 1.10 for Part 1 (when $k \leq n - 3$, i.e., $t \geq 3$). By Claim 7, $y_ty_1 \in D$. Without loss of generality, we may assume that x_1y_1 and $y_tx_2 \in D$ since $\lambda = 1$. Then from Claims 8, 9 and 10 it follows that

$$A(R \to \{x_3, x_4, \dots, x_k, x_1\}) = A(\{x_3, x_4, \dots, x_k\} \to R) = \emptyset.$$

From this and

$$d^-(x_2, \{y_1, y_2, \dots, y_{t-1}\}) = d^+(x_1, \{y_2, y_3, \dots, y_t\}) = 0$$

we obtain that x_1, y_2 and x_1, y_t are two distinct pairs of nonadjacent vertices and $d(y_2, C) \leq 1$, $d(y_t, C) \leq 2$, $d(x_1, R) = 1$. Therefore $d(y_2) \leq n-k+2$, $d(y_t) \leq n-k+2$ (by (2)) and $d(x_1) \leq 2k-1$. The last three inequalities imply that $d(y_2) + d(x_1) \leq 2n - 2$ and $d(y_t) + d(x_1) \leq 2n - 2$, which contradicts Lemma 5 and completes the discussion of Part 1.
Part 2. $k = n - 2$, i.e., $t = 2$.
For this part first we will proved Claims 11-16 below.

Claim 11. *If $x_iy_f \in D$ and $y_2y_1 \notin D$, where $i \in [1, n-2]$ and $f \in [1, 2]$, then there is no $l \in [3, n-2]$ such that $y_fx_{i+l-1} \in D$ and $d(y_f, \{x_{i+1}, x_{i+2}, \dots, x_{i+l-2}\}) = 0$.*

Claim 12. *$y_2y_1 \in D$ (i.e., if $k = n - 2$, then $\langle V(D) - V(C)\rangle$ is strong).*

For the proof Claim 12 we consider the following cases.

Case 12.1. $d^+(y_1, C[x_3, x_{n-2}]) \geq 1$.

Case 12.2. $d^+(y_1, C[x_3, x_{n-2}]) = 0$.

Then $d^+(y_1, C[x_2, x_{n-2}]) = 0$ and either $y_1x_1 \in D$ or $y_1x_1 \notin D$.

Subcase 12.2.1. $y_1x_1 \in D$.

Subcase 12.2.2. $y_1x_1 \notin D$.

Claim 13. *For any $i \in [1, n-2]$ and $f \in [1, 2]$ the following holds*
i) $d^-(y_f, \{x_{i-1}, x_i\}) \leq 1$ and ii) $d^+(y_f, \{x_{i-1}, x_i\}) \leq 1$.

Claim 14. *If $x_iy_f \in D$ (respectively, $y_fx_i \in D$), then $d(y_f, \{x_{i+2}\}) \neq 0$ (respectively, $d(y_f, \{x_{i-2}\}) \neq 0$), where $i \in [1, n-2]$ and $f \in [1, 2]$.*
We assumed that $n \geq 6$ and considered the following cases.

Case 14.1. $A(R \to \{x_3, x_4, \dots, x_{n-3}\}) \neq \emptyset$.

Case 14.2. $A(R \to \{x_3, x_4, \dots, x_{n-3}\}) = \emptyset$.

Subcase 14.2.1. $y_2x_{n-2} \in D$.

Subcase 14.2.2. $y_2x_{n-2} \notin D$ *and* $y_2x_1 \in D$.

Subcase 14.2.3. $y_2x_{n-2} \notin D$ *and* $y_2x_1 \notin D$.

Claim 15. *If $x_iy_f \in D$ and the vertices y_f, x_{i+1} are nonadjacent, then the vertices x_{i+1}, y_{3-f} are adjacent, where $i \in [1, n-2]$ and $f \in [1, 2]$.*

Claim 16. *If $x_iy_j \in D$, where $i \in [1, n-2]$ and $j \in [1, 2]$, then $y_jx_{i+2} \in D$.*

Case 16.1. $y_2x_3 \in D$.

Case 16.2. $y_2x_3 \notin D$.

We will now complete the proof of Theorem by showing that D is isomorphic to $K^*_{n/2,n/2}$. Without loss of generality, we assume that $x_{n-2}y_1 \in D$. Then using Claims 12, 13, 14 and 16 we conclude that y_1, x_1 are nonadjacent (Claim 13), $y_1x_2 \in D$ (Claim 16), $x_1y_2, y_2x_1 \in D$ (Claim 12), x_2, y_2 also are nonadjacent (Claim 13), $y_2x_3 \in D$ (Claim 16) and $x_2y_1 \in D$ (Claim 12). By continuing this procedure, we eventually obtain that n is even and

$$N^+(y_1) = N^-(y_1) = \{y_2, x_2, x_4, \dots, x_{n-2}\} \quad \text{and}$$

$$N^+(y_2) = N^-(y_2) = \{y_1, x_1, x_3, \dots, x_{n-3}\}.$$

If $x_ix_j \in D$ for some $x_i, x_j \in \{x_1, x_3, \dots, x_{n-3}\}$, then clearly $|C[x_i, x_j]| \geq 5$ and $x_ix_jx_{j+1} \dots x_{i-1}y_1x_{i+1} \dots x_{j-2}y_2x_i$ is a cycle of length $n-1$, contrary to our assumption. Therefore $\{y_1, x_1, x_3, \dots, x_{n-3}\}$ is a independent set of vertices. For the same reason $\{y_2, x_2, x_4, \dots, x_{n-2}\}$ also is a independent set of vertices. Now from condition A_0 it follows that D is isomorphic to $K^*_{n/2,n/2}$. This completes the proof of Theorem 9. □

5. CONCLUDING REMARKS

A Hamiltonian bypass in a digraph is a subdigraph obtained from a Hamiltonian cycle of D by reversing one arc.
Using Theorem 9, the first author have proved that if a strong digraph D of order $n \geq 4$ satisfies the condition A_0, then D contains a Hamiltonian bypass or D is isomorphic to one tournament of order 5.

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