Some Properties of Hadamard Matrices

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ABSTRACT

Hadamard transform is an important tool for the investigation of some problems of Quantum Computing, Coding Theory and Cryptology, Statistics, Image Analysis, Signal Processing, Fault-Tolerant Systems, Analysis of Stock Market Data, Combinatorial Designs and so on. Here we present one numerical property of Hadamard matrices.

Keywords

Hadamard matrices, Sylvester (Walsh) matrices

1. INTRODUCTION

There are various types of matrices in the literature having distinct properties useful for numerous applications, both practical and theoretical. The famous matrix with orthogonal property is a Hadamard matrix, which was first defined by J.J. Sylvester in 1867 and was studied further by Hadamard in 1893. A Hadamard matrix has a simple structure and is a square matrix whose entries are either +1 or −1 and the rows are mutually orthogonal. In geometric terms, this means that every two different rows in a Hadamard matrix represent two orthogonal vectors. This definition implies that the corresponding properties hold for columns as well. Hadamard matrices are used to compute the Hadamard transform (also known as the Walsh-Hadamard transform), which has plenty of practical applications. It has applications even in Banach space theory (see e.g. [1-4]).

The most important open question in the theory of Hadamard matrices is the question of existence. Hadamard conjectured that the Hadamard matrix of order $4k$ exists for every positive integer k. Despite the efforts of several mathematicians, this conjecture remains unproved even though it is widely believed that it is true. This condition is necessary, while the sufficiency part is still open. The smallest order for which no Hadamard matrix is presently known is 668.

The goal of this communication is to consider some numerical functionals of the Hadamard and Sylvester matrices and their estimations. The communication is based on the results of the papers [4] and [5].

2. THE CASE OF HADAMARD MATRICES

Taking into account the problems related to Hadamard conjecture, let us denote by $\mathbb{N}_{\mathcal{H}}$ the set of all positive integers n for which there exists a Hadamard matrix of order n. Let $\mathcal{H}_n^{all}$ be the set of all Hadamard matrices of order $n, n \in \mathbb{N}_{\mathcal{H}}$. Fix a number $p \geq 1$ and for a Hadamard matrix $H_n = [h_{ik}^n]$ consider the following numerical functionals:

$$\varrho_{p,H_n}(m) = (\textstyle\sum_{k=1}^n |\sum_{i=1}^m h_{ik}^n|^p)^{1/p}, \quad m = 1,2,\cdots,n,$$

$$\varrho_{p,H_n} = \max_{1\leq m\leq n} \varrho_{p,H_n}(m) \text{ and } \varrho_{p,n} = \max_{H_n\in\mathcal{H}_n^{all}} \varrho_{p,H_n}.$$

It is obvious that $n^{1/p} \leq \varrho_{p,H_n} \leq n^{\left(1+\frac{1}{p}\right)}$ for every $p \geq 1$ and any $H_n \in \mathcal{H}_n^{all}$. Our aim is to improve the last estimations. The following statement gives more precise upper and lower bownds.

Theorem 2.1. *For every $p \geq 1$ and any $n \in \mathbb{N}_{\mathcal{H}}$ we have*

$$\frac{1}{\sqrt{2}} \cdot n^{(p+2)/2p} \leq \varrho_{p,n} \leq n^{(p+2)/2p}, \quad for \quad 1 \leq p \leq 2,$$

and

$$\varrho_{p,n} = n, \quad for \quad p \geq 2.$$

In the proof of Theorem 2.1 the probabilistic methods in functional spaces are used.

The following statement is a simple consequence of Theorem 2.1.

Corollary 2.2. *For any $n \in \mathbb{N}_{\mathcal{H}}$ we have*

$$\frac{1}{\sqrt{2}} \cdot n^{3/2} \leq \varrho_{1,n} \leq n^{3/2}.$$

3. THE CASE OF SYLVESTER MATRICES

Particular case of the Hadamard matrix is the Walsh matrix, which is defined by the following recursive formula:

$$S^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S^{(n)} = \begin{bmatrix} S^{(n-1)} & S^{(n-1)} \\ S^{(n-1)} & -S^{(n-1)} \end{bmatrix}, \ n = 2,3,\cdots. \quad (3.1)$$

Evidently $S^{(n)}$ is a square matrix of order 2^n. This construction was studied by Joseph L. Walsh in 1923. It should be noted that in fact, examples of Hadamard matrices were actually first constructed earlier by James Joseph Sylvester in 1867. He had noted that if H is a Hadamard matrix of order n, then the block-matrix

$$\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

is a Hadamard matrix of order $2n$. This observation can be applied repeatedly and it leads to the sequence (3.1) of matrices, also called as Sylvester matrices.

Let $S^{(n)} = \left[s_{ik}^{(n)}\right]$ be a Sylvester (Walsh) matrix of order 2^n. Fix a number $p \geq 1$ and consider the following functionals

$$\varrho_p^{(n)}(m) = \left(\sum_{k=1}^{2^n} \left|\sum_{i=1}^m s_{ik}^{(n)}\right|^p\right)^{1/p}, \quad m = 1,2,\cdots,2^n$$

and

$$\varrho_p^{(n)} = \max_{1\leq m\leq 2^n} \varrho_p^{(n)}(m).$$

By the analogy of the Hadamard matrices it is easy to see that for any positive integer n and a number $p \geq 1$ we have

$$2^{n/p} \leq \varrho_p^{(n)} \leq 2^{n\left(1+\frac{1}{p}\right)}.$$

As the Sylvester matrix is a Hadamard matrix, then using Theorem 2.1 we can slightly improve the last inequality:

$$2^{n/p} \leq \varrho_p^{(n)} \leq 2^{n\left(\frac{1}{2}+\frac{1}{p}\right)}.$$

In case $p = 1$ the following exact equality for the functional $\varrho_1^{(n)}$ is valid:

Theorem 3.1. *For every positive integer n we have*

$$\varrho_1^{(n)} = \max_{1\le m\le 2^n} \varrho_1^{(n)}(m) = \frac{3n+7}{9}\cdot 2^n + (-1)^n \cdot \frac{2}{9}.$$

For any n the maximum is attained at the points $m_n = \frac{2^{n+1}+(-1)^n}{3}$ and $m_n' = \frac{5\cdot 2^{n-1}+(-1)^{n-1}}{3}$.

4. UNSOLVED PROBLEM

Let us formulate the assertion of Theorem 3.1 in the following manner:

$$\varrho_1^{(n)} = \sum_{k=1}^{2^n} \left|\sum_{i=1}^{m_n} s_{ik}^{(n)}\right|,$$

where $m_n = \frac{2^{n+1}+(-1)^n}{3}$.

Now let us consider a permutation $\sigma: \{1,2,\cdots,2^n\} \to \{1,2,\cdots,2^n\}$ and the following expression:

$$\sum_{k=1}^{2^n} \left|\sum_{i=1}^{m_n} s_{\sigma(i)k}^{(n)}\right|.$$

By Corollary 2.2 for every permutation $\sigma: \{1,2,\cdots,2^n\} \to \{1,2,\cdots,2^n\}$ we have

$$\sum_{k=1}^{2^n} \left|\sum_{i=1}^{m_n} s_{\sigma(i)k}^{(n)}\right| \le 2^{3n/2}.$$

The authors do not know yet the answer for the following conjecture:

Conjecture. *For any positive integer n and for any permutation $\sigma: \{1,2,\cdots,2^n\} \to \{1,2,\cdots,2^n\}$ the following inequality holds:*

$$\sum_{k=1}^{2^n} \left|\sum_{i=1}^{m_n} s_{\sigma(i)k}^{(n)}\right| \ge \frac{3n+7}{9}\cdot 2^n + (-1)^n \cdot \frac{2}{9}.$$

Note that we have conducted a lot of computer experiments. The results do not contradict this Conjecture, though a theoretical proof is not known yet.

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