Open Problems in Gossip/Broadcast Schemes and the Possible Application of the Method of Local Interchange *

Vilyam, Hovnanyan
Institute for Informatics and Automation Problems
NAS of Armenia, 0014 Yerevan, Armenia
e-mail: williamhovnanyan@gmail.com

Vahagn, Poghosyan
Institute for Informatics and Automation Problems
NAS of Armenia, 0014 Yerevan, Armenia
e-mail: povahagn@gmail.com

Suren, Poghosyan
Institute for Informatics and Automation Problems
NAS of Armenia, 0014 Yerevan, Armenia
e-mail: psuren55@yandex.ru

ABSTRACT
The gossip problem (telephone problem) is an information dissemination problem in which each of the n nodes of communication network has a unique piece of information that should be transmitted to all other nodes using two-way communications (telephone calls) between the pairs of nodes. During a call between the given two nodes, they exchange the whole information known to them at that moment. In this paper the method of *local interchange* is introduced to investigate gossip problems. This method is based on a repetitive use of *permute higher* and *permute lower* operations, which map one gossip graph with n vertices to another by moving only its edges without changing the labels of edges (the moments of corresponding calls). Using this operation we obtained results some of which are new and the others prove already the known ones in completely new way. However, the main topic of this paper is some open problems on the applications of this method.

Keywords
Networks, telephone problem, gossip problem, method of local interchange

1. INTRODUCTION
Gossiping is one of the basic problems of information dissemination in communication networks. The gossip problem (also known as a telephone problem) is attributed to A. Boyd (see ex. [4] for review), although to the best knowledge of the reviewers, it was first formulated by R. Chesters and S. Silverman (Univ. of Witwatersrand, unpublished, 1970). Consider a set of n persons (nodes) each of which initially knows some unique piece of information that is unknown to the others, and they can make a sequence of telephone calls to spread the information. During a call between the given two nodes, they exchange the whole information known to them at that moment. The problem is to find a sequence of calls with minimum length (minimal gossip scheme), by which all the nodes will know all pieces of information (complete gossiping). It has been shown in numerous works [1–4] that the minimal number of calls is $2n-4$ when $n \geq 4$ and $1, 3$ for $n = 2, 3$, respectively. Since then many variations of gossip problem have been introduced and investigated [5–8, 10].

Another variant of the Gossip problem can be formulated by considering the minimum amount of time required to complete gossiping among n persons, where the calls between non-overlapping pairs of nodes can take place simultaneously and each call requires one unit of time [11, 12, 17].

Obviously, the gossip problem can be easily modeled as a graph, whose vertices represent people in gossip scheme and edges represent calls between them (each of them has weight which represents the moment when communication took place). So the graph is called a complete gossip graph if there are ascending paths between all the pairs of vertices of the graph.

In section 2 we introduce the method of *local interchange*, which is based on a repetitive use of *permute higher* and *permute lower* operations. These operations locally act on the adjacent edges to a given edge in a gossip graph (a non-local *interchange of two vertices from some point onwards in a list of telephone calls* is defined in [4]). In section 3 the method of *local interchange* is demonstrated for the alternative derivation of the minimum number of calls in gossip scheme. Another application of *local interchange* is described in section 4, where the procedure for obtaining minimum time gossip graphs in which no one hears his own information (NOHO graphs, [19, 21]) from gossip graphs based on Knödel graphs is found. Since the minimal time of gossip graphs is known [17], this procedure allowed us to establish the value of minimal time for NOHO graphs $T = \lceil \log n \rceil$. Then we present a new use case of these operations concerning the construction of Gossip graphs providing a full information exchange with minimal number of calls in minimum time. We also describe the variations of gossip scheme depending on the size of a basis. Finally, in the last section we define some open problems on this method and suggest our hypothesis for them.

2. DESCRIPTION OF THE METHOD
A gossip scheme (a sequence of calls between n nodes) can be represented by an undirected edge-labeled graph $G = (V, E)$ with $|V(G)| = n$ vertices. The vertices and

*The main results that we obtained are presented in [26] and [27]. Here we shortly introduce the operation, results and define some open problems for further research.

edges of G represent correspondingly the nodes and the calls between the pairs of nodes of a gossip scheme. Such graphs may have multiple edges, but not self loops. An edge-labeling of G is a mapping $t_G : E(G) \to Z^+$. The label $t_G(e)$ of a given edge $e \in E(G)$ represents the moment of time, when the corresponding call occurs.

A sequence $L = (v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ with vertices $v_i \in V(G)$ for $0 \le i \le k$ and edges $e_i \in E(G)$ for $1 \le i \le k$ is called a walk of length k from a vertex v_0 to a vertex v_k in G, if each edge e_i joins two vertices v_{i-1} and v_i for $1 \le i \le k$. A walk, in which all the vertices are distinct is called a path. If $t_G(e_i) < t_G(e_j)$ for $1 \le i < j \le k$, then P is an ascending path from v_0 to v_k in G. Given two vertices u and v, if there is an ascending path from u to v, then v receives the information of u.

Note that two non-adjacent edges can have the same label. Since we consider only (strictly) ascending paths, then such edges (i.e. calls) are independent, which means that during the consideration of minimal number of calls the edges with the same label can be reordered arbitrarily but for any $t_1 < t_2$ all the edges with the label t_1 are ordered before any of the edges with the label t_2. Therefore in this case we can assume that there are no any two edges e_1 and e_2 such that $t_G(e_1) = t_G(e_2)$.

Denote the set of edges adjacent to a given vertex v by $E_v(G)$. Given an edge e and one of its two endpoints v we consider following two subsets of the set $E_v(G)$:

$$\rho_v^+(e, G) = \{e' \in E_v(G) | t_G(e') \ge t_G(e)\}, \qquad (2.1)$$

$$\rho_v^-(e, G) = \{e' \in E_v(G) | t_G(e') \le t_G(e)\}. \qquad (2.2)$$

Sometimes we omit the argument G in notations.

Definition 1. *An identification of two vertices v_1 and v_2 [4] in a gossip graph G is a gossip graph G', which is defined as follows: The edges between the vertices v_1 and v_2 are deleted and these two vertices are replaced by a vertex u, whose set of incident edges is $E_u(G') = E_{v_1}(G) \cup E_{v_2}(G)$*

An interchange of two vertices in a calling scheme is defined as follows: started from the indicated moment of a time to the end of the calling scheme two vertices (and the edges adjacent to them) are replaced by each other. In this paper we define the concept of local interchange - an interchange that is defined only for the adjacent vertices.

Definition 2. *The permute higher operation $P^+(e)$ on a selected edge $e \in E(G)$ connecting vertices u and v is called a modification of G, which moves the edges of G adjacent to e as follows:*

$$E_u(P^+(e)G) = \rho_u^-(e, G) \cup \rho_v^+(e, G), \qquad (2.3)$$

and

$$E_v(P^+(e)G) = \rho_v^-(e, G) \cup \rho_u^+(e, G). \qquad (2.4)$$

Correspondingly, the permute lower operation $P^-(e)$ moves the edges of G adjacent to e as follows:

$$E_u(P^-(e)G) = \rho_u^+(e, G) \cup \rho_v^-(e, G), \qquad (2.5)$$

and

$$E_v(P^-(e)G) = \rho_v^+(e, G) \cup \rho_u^-(e, G). \qquad (2.6)$$

The operators P^+ and P^- are called the *operators of local interchange*

Lemma 1. *The result of the action of the operators of local interchange on a complete gossip graph is also a complete gossip graph.*

Actually, if a call between two participants takes place at time unit t_0, started from that point they both have the same information and are equivalent, hence if we change all the calls which will take place after that point (permute higher) our new gossip scheme would also be complete. At the same time, we changed neither the number of edges, nor the number of rounds required to perform gossiping. The same assumptions could be made in case of permute lower operation.

Let us define an operation on gossip graphs $A^+ = (P^+(e_1)P^+(e_2)\ldots P^+(e_p))$ $(A^- = (P^-(e_1)P^-(e_2)\ldots P^-(e_p)))$ is the sequence of permute higher (lower) operations on edges e_i, $i = 1, \ldots, p$.

Lemma 2. *The result of the operation A^+ (A^-) does not depend on the order of the edges e_i.*

For the detailed proof of the lemma see [26].

So, in this section the operations of local interchange and their main properties were described.

3. APPLICATIONS OF THE METHOD OF LOCAL INTERCHANGE

In this section we are going to present some details on the application of the method of local interchange. For the sake of brevity we will omit details of the proofs or even proofs completely for some theorems.

Theorem 1. *The minimum required number of calls in complete gossip schemes is equal to $2n - 4$, $n \ge 4$.*

Proof. As already mentioned in the introduction it has been shown in numerous works ([1–4]) that the minimal number of calls in gossip schemes is $2n4$, where n is the number of vertices. In this section we will prove this statement in completely new way, which is based on the operations of local interchange.

Let us denote by $f(n)$ the minimum number of edges required to construct gossip graph on n vertices. There are many solutions of this problem that give the number of edges (calls) equal to $2n-4$. Therefore, $f(n) \le 2n-4$ is valid. So the important part is to prove that $f(n) \ge 2n - 4$ also.

Consider a gossip graph with n vertices and the number of edges equal to $f(n)$. Our goal is to prove that

$f(n) \geq f(n-1)+2$. After that, by taking into account that $f(4)=4$ we will get that $f(n) \geq 2n-4$.

Suppose we have a graph G with $f(n)$ calls. There are 3 possible kinds of graphs depending on repetition of information.

1) It contains a cycle.

2) There are no cycles (NOHO graphs).

3) There are no duplicates of information (NODUP graphs).

For each of these scenarios the detailed explanation is brought in [26]. □

In [19] it is mentioned that the minimum time needed to complete gossiping is $\lceil \log n \rceil$. In [22] a problem of finding the minimum time of gossiping in case of NOHO graphs (problem 26) is raised. Here we will show a new method of construction of NOHO graphs with minimum gossiping time by applying an operation of local interchange on Knödel graphs ([7]).

Definition 3. The Knödel graph on $n \geq 2$ vertices (n even) and of degree $\Delta \geq 1$ is denoted by $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the pairs (i,j) with $i = 1,2$ and $0 \leq j \leq n/21$. For every j, $0 \leq j \leq n/21$ and $l = 1,\ldots,\Delta$, there is an edge with the label l between the vertex $(1,j)$ and $(2,(j+2^{l1}1) mod n/2)$.

From the definition it follows that this graph is connected only if $\Delta \geq 2$. We will consider the Knödel graphs with $\Delta = \lceil \log_2 n \rceil$.

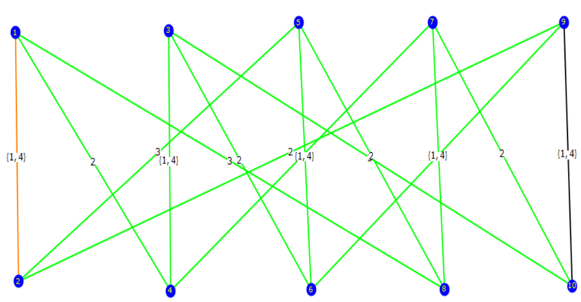

Fig. 1. Gossip graph based on Knödel graph

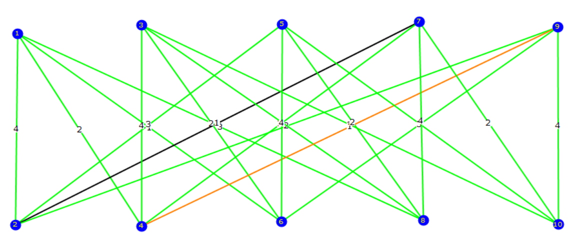

Fig. 2. NOHO graph based on Knödel graph

In [18] it was shown that the combination of two Knödel graphs with $\Delta_1 = \lceil \log_2 n \rceil$ and $\Delta_2 = 1$ is a complete gossip graph (see Fig. 1). Let us define operation $A_l^- = \{P^-(e) | t_G(e) < t_G(l)\}$ as an operation of permutation of all edges that have lower weight value (or label) then l. After application of A_2^- on Knödel graphs all edges with label 1 will be permuted(see Fig. 2).

Lemma 3. The result of application of A_2^- on complete gossip graph based on Knödel graph is a NOHO gossip graph with minimum gossiping time $T = \lceil \log_2 n \rceil$.

For the proof of this lemma see [26].

A new use case of the operations of local interchange is going to be introduced here. It helps us to construct a gossip scheme with a minimum number of edges and minimum gossiping time (w.l.o.g. minimum gossip scheme/graph). Particularly, we will use them to obtain a canonical form of basis for minimum gossip scheme.

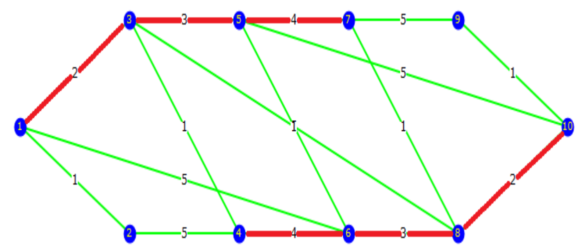

Fig. 3 NOHO graph.

Here we want to present different ways of construction of the minimum gossip schemes. In [23] and [8] it was shown that the time required to perform gossiping with minimum number of calls and minimum time (in other words - with minimum number of rounds) is at least $2\lceil \log_2 n \rceil - 3$. The methods that allow to construct minimum gossip schemes were also considered that have an underlying basis (a minimum gossip graph) for gossiping, and the new vertices participate in gossiping by forming the attached trees which are connected to basis. It is obvious, that if the basis has a minimum number of edges, then the full graph also would have the same. Hence, the main problem here is to perform gossiping in minimum time. In other words, the time in which the basis performs gossiping should be minimal and the corresponding attached trees should not affect it. So, our goal is to construct such gossip schemes with variety of size of the basis. As was mentioned

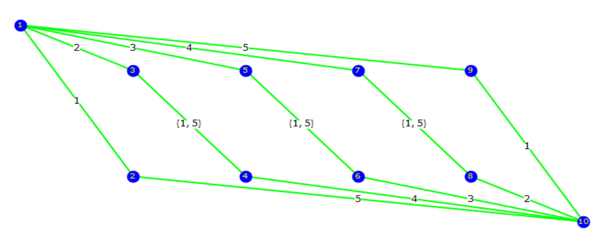

Fig. 4. Minimum gossip graph in "canonical" form

above, we will start from the application of operation A^+ on the NOHO graphs.

The application of the operations $A^+(e_1, e_2, \dots, e_p)$ and $A^+(e'_1, e'_2, \dots, e'_p)$ on NOHO graphs (Fig. 3) gives as new "canonical" form of minimum gossip graph, which is a more convenient form of basis (Fig. 4). Here $p = 3$ and the edges e_1, e_2, e_3 (e'_1, e'_2, e'_3) are highlighted in red.

After this transformation the obtained graph will be used as a basis with size $k = n'/2$, where n' is the number of vertices of the basis. Now consider the new set of vertices with the same size that communicates with the given set by incoming and outgoing calls. Let the number of rounds by which such a communication takes place be denoted by r. Obviously, the number of rounds to perform gossiping will be at least $2r + k$. This estimate will be reached only if the attached trees would not affect the gossiping process of the basis. It means, that the process of involving new vertices in gossiping should be in parallel with the main gossiping process of the basis. Hence, we have to modify the form of the basis such that we could make in-calls and out-calls with the attached trees in process of gossiping of the basis. For that, we should reorder the calls of basis, in such a way that it will be possible to make such in-calls and out-calls. In other words the calls between two parts of gossip scheme should have an increasing and decreasing order. An additional note is that depending on r the number of attached vertices increases exponentially. Such gossiping process for $n = 24$ is demonstrated in Fig. 5, where $r = 2$ and $k = 3$.

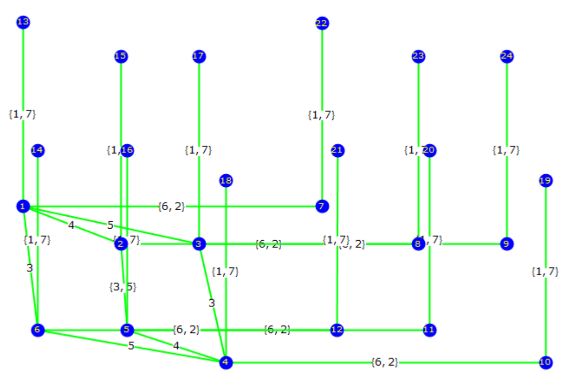

Fig. 5. Minimum gossip graph $(n = 24, k = 3, r = 2)$.

Obviously, the number of rounds required to perform gossiping is $T' = 2r + k$ which coincides with the minimum possible number of rounds for some values of n, but this scheme is not the only possible one to perform gossiping in minimum time. By changing the size of the basis (k) and number of rounds of the "batch" calls (r) a new minimum gossiping scheme will be obtained (see Fig. 6).

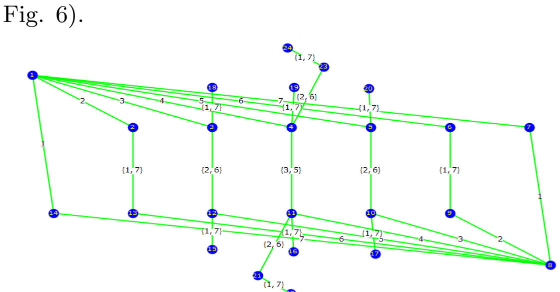

Fig. 6. Minimum gossip graph $(n = 24, k = 7, r = 0)$

From these examples it is obvious that in each round new vertices are included in communication, but the number of these vertices is limited. So, taking this into account the following lemma could be formulated.

Lemma 4. The number of vertices in minimum gossip graph with the size of basis k and the number of rounds of the "batch" calls r is limited by the following estimates:

$$n \leq 2^{k/2+r+1}, \text{ if } k \text{ is even}, \tag{3.1}$$

$$n \leq 3 \times 2^{(k-1)/2+r}, \text{ if } k \text{ is odd}. \tag{3.2}$$

For the proof of this lemma see [27].

In case of even k it is possible by changing the structure and gossiping time of basis to reach more vertices involved in gossiping.

Now let us consider the number of rounds which are necessary to perform gossiping with minimum number of calls. Let it be denoted by T. As it was already mentioned $T = 2\lceil \log_2 n \rceil - 3$. Our goal is to show the dependency between T and n depending on the construction of gossiping scheme. In other words, we offer the method of construction of minimum gossip graphs for the particular values of n, where the values of k and r can vary. Let T' be the minimum number of rounds of the gossip scheme with $2n - 4$ calls, the size of the basis k and the number of rounds of "batch" calls r.

Theorem 2. In gossiping scheme with the size of the basis k and the number of rounds of the "batch" calls r the number of rounds necessary to complete gossiping among n nodes is:

$$T' = 2\lceil \log_2 n \rceil - 2, \text{ if } k \text{ is even}, \tag{3.3}$$

$$T' = 2\left\lceil \log_2 \frac{n}{3} \right\rceil + 1, \text{ if } k \text{ is odd}. \tag{3.4}$$

For the proofs of all the lemmas and theorems as well as for better understanding other omitted details, please refer to [26] and [27] before continuing.

4. OPEN PROBLEMS IN GOSSIP/BROADCAST SCHEMES

In the current section we are going to raise some open problems in gossip/broadcast problem sphere that we find the problems of interest in regards to this method.

Open Problem 1. Sufficient and necessary conditions for the application of the methods of local interchange on the k fault-tolerant gossip graph to not affect the level of the fault-tolerance of the graph.

Remark 1. We have shown in [18] that in the case of $k = 0$ (complete gossip graph) the application of this method on the graph still yields another gossip graph. On the other side by using software tool designed by us *and described in [28] we have noted that it often does not affect the level of fault-tolerance of Knödel based k fault-tolerant graph. But it does affect the level of fault-tolerance of the fault-tolerant graphs that are constructed by graph combination method (see [29]). So we find this problem the problem of interest as it would yield different variations for the structure of fault-tolerant gossip graph.*

Along with gossip problems there is another class of problems that are based on the process of broadcasting. Here the main difference from gossiping is that the originator of message is only one vertex and its message should be transmitted to the other nodes. So, a minimal k-falut-tolerant broadcast scheme is the communication scheme on n nodes where the originator can broadcast its message in spite of k arbitrary faults in communcation lines in optimal time $T_n(k)$. So, let us denote by $B_k(n)$ the minimum number of communication lines of any minimal k fault-tolerant broadcast scheme on n nodes. Note here we speak about communication lines and not calls as in case of gossiping (a line can contain multiple calls). The best known lower bound on $T_k(n)$ is $\lceil \log_2 n \rceil + k$ for general values of k and n.

Open Problem 2. *Is it possible to use the method of local interchange to obtain k fault-tolerant minimal broadcast schemes.*

Remark 2. *With the help of Graph Plotter (see [28]) we have noted that the application of* A_2^- *on the* $G = W_{\log_2 n,n} + W_{1,n}$ *yields minimal broadcast* 1 *fault-tolerant scheme with* $T = \log_2 n + 1$.

Assuming that the calls between non-overlapping pairs of nodes can take place simultaneously, the minimum amount of time $T(n)$ required to complete gossiping is $\lceil \log_2 n \rceil$ for even n and $\lceil \log_2 n \rceil + 1$ for odd n. As it is shown in [8], any gossip scheme on $n \geq 4$ vertices with $2n - 4$ calls has at least $2 \lceil \log_2 n \rceil - 3$ rounds.

Open Problem 3. *Is it possible by the help of this method find the minimum number of calls of a gossip scheme with time (rounds) T(n).*

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