Description of Stable Subsets of n-Dimensional Multivalued Discrete Torus

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ABSTRACT
The n-dimensional torus with generating cycles of even length is considered. Stable subsets of the torus are determined and described.

Keywords
Discrete torus, standard arrangement, stable subset

1. INTRODUCTION

<u>Definition 1.</u> For any integers $1 \le k_1 \le k_2 \le \cdots \le k_n < \infty$ ***the multivalued n-dimensional torus*** $T^n_{k_1k_2\cdots k_n}$ has been defined as the set of vertices: $T^n_{k_1k_2\cdots k_n} = \{(x_1, x_2, \cdots, x_n) / -k_i + 1 \le x_i \le k_i, x_i \in Z, 1 \le i \le n\}$, where two vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ of $T^n_{k_1k_2\cdots k_n}$ are considered as neighbours, if they differ by exactly one coordinate for which either $| x_i - y_i | = 1$ or the values equal $(-k_i + 1)$ and k_i respectively. The sum and difference of these vectors has been defined in the following way: $z = x \pm y = (x_1 \pm y_1, x_2 \pm y_2, \cdots, x_n \pm y_n) = (z_1, z_2, \cdots, z_n)$, where $-k_i + 1 \le z_i \le k_i$ and $z_i \equiv (x_i \pm y_i)(\mathrm{mod}\ 2k_i)$.

We denote by $\| x \|$ the ***norm*** of a vertex $x = (x_1, x_2, \ldots, x_n)$ where $\| x \| = \sum_{i=1}^{n} | x_i |$, and denote by $\rho(x, y)$ ***the distance between the vertices*** $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ where $\rho(x, y) = \| x - y \|$.

The set $S^n(x,k) = \{ y \in T^n_{k_1k_2\cdots k_n} / \rho(x, y) \le k \}$ is called a ***sphere*** with the centre $x \in T^n_{k_1k_2\cdots k_n}$ and radius k, and the set $O^n(x,k) = \{ y \in T^n_{k_1k_2\cdots k_n} / \rho(x, y) = k \}$ is the ***envelope*** with centre x and radius k.

Let $e_i = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ denote ***the unit vector of i-th direction***, where $\alpha_i = 1$ and $\alpha_j = 0$ for $j \ne i$, and let $\tilde{1}$ and $\tilde{0}$ be the vectors with all 1 and all 0 coordinates respectively: $\tilde{1} = (1,1,\cdots,1)$ and $\tilde{0} = (0,0,\cdots,0)$.

For any subset $A \subseteq T^n_{k_1k_2\cdots k_n}$ and any i $(1 \le i \le n)$ and j $(-k_i + 1 \le j \le k_i)$ we make the following designation:

$$A + je_i = \{ x + je_i \ / \ x \in A \}.$$

We will consider partition of $T^n_{k_1k_2\cdots k_n}$ (respectively partition of $A \subseteq T^n_{k_1k_2\cdots k_n}$) on i-th direction, $1 \le i \le n$ and j-th value, $-k_i + 1 \le j \le k_i$ and will denote by $T^n_i(j)$ (respectively by $A_i(j)$):

$$T^n_i(j) = \{x = (x_1, x_2, \cdots x_n) \in T^n_{k_1k_2\cdots k_n} \ / \ x_i = j\},$$

$$A_i(j) = \{x = (x_1, x_2, \cdots x_n) \in A \ / \ x_i = j\} = A \bigcap T^n_i(j).$$

Notice that the intersections of the sphere $S^n(x,k)$ and the envelope $O^n(x,k)$ with the $(n-1)$-dimensional torus $T^n_i(x_i + j)$, are respectively the sphere and envelope with the centre $x + je_i$ and radius $k - | j |$ in $T^n_i(x_i + j)$. We make the following designations:

$$S^n_i(x + je_i, k - | j |) = \{y \in S^n(x, k) \ / \ y_i = x_i + j\} = S^n(x,k) \bigcap T^n_i(x_i + j);$$

$$O^n_i(x + je_i, k - | j |) = \{y \in O^n(x, k) \ / \ y_i = x_i + j\} = O^n(x,k) \bigcap T^n_i(x_i + j),$$

where in case of $k - | j | < 0$ these sets are empty: $S^n_i(x + je_i, k - | j |) = O^n_i(x + je_i, k - | j |) = \varnothing$.

It is clear that $T^n_{k_1k_2\cdots k_n} = \bigcup_{j=-k_i+1}^{k_i} T^n_i(j)$, $A = \bigcup_{j=-k_i+1}^{k_i} A_i(j)$,

$$S^n(x,k) = \bigcup_{j=-k_i+1}^{k_i} S^n_i(x + je_i, k - | j |),$$

$$O^n(x,k) = \bigcup_{j=-k_i+1}^{k_i} O^n_i(x + je_i, k - | j |),$$

for each i, $1 \le i \le n$;

<u>Definition 2.</u> For a given subset $A \subseteq T^n_{k_1k_2\cdots k_n}$ we say that a vertex $x \in A$ is an ***interior point*** of A, if all its neighbouring vertices belong to A. Otherwise $x \in A$ is called ***a boundary vertex*** of A. We denote by $B(A)$ and $\Gamma(A)$, respectively, the subset of all interior and boundary points of A.

For each $A_i(j)$ in the partition of $A=\bigcup_{j=-k_i+1}^{k_i} A_i(j)$ we denote by $B(A_i(j))$ and $\Gamma(A_i(j))$, respectively, the subsets of its interior and boundary vertices in $(n-1)-$ dimensional torus $T_i^n(j)$.

For any vertex $x=(x_1,x_2,\ldots,x_n)$ of $T^n_{k_1k_2\cdots k_n}$, we denote by $|x|$ and $\delta(x)$ the vectors $|x|=(|x_1|,|x_2|,\ldots,|x_n|)$ and $\delta(x)=(\alpha_1,\alpha_2,\ldots,\alpha_n)$, where $\alpha_i=1$ for $x_{n-i+1}>0$ and $\alpha_i=0$ for $x_{n-i+1}\le 0$.

In general, for n-dimensional vectors $x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_n)$ with nonnegative integer coordinates, we say that the vector x lexicographically precedes y (written by $x\prec y$), if there is a number r, $1\le r\le n$, such that $x_i=y_i$ for $1\le i<r$ and $x_r<y_r$.

Now we order the vertices of the torus $T^n_{k_1k_2\cdots k_n}$ as follows:

vertex x precedes vertex y (written by $x\Leftarrow y$), if and only if

1. $\|x\|<\|y\|$ or
2. $\|x\|=\|y\|$ and $\delta(y)$ lexicographically precedes $\delta(x)$, or
3. $\|x\|=\|y\|$, $\delta(x)=\delta(y)$ and $|y|$ lexicographically precedes $|x|$.

It is easy to check that this ordering between the vertices of the torus $T^n_{k_1k_2\cdots k_n}$ is a linear order.

Definition 3. The first a vertices of the torus $T^n_{k_1k_2\cdots k_n}$ by the above determined liner order we call ***standard arrangement*** of cardinality a, $0\le a\le |T^n_{k_1k_2\cdots k_n}|$.

Torus $T^n_{k_1k_2\cdots k_n}$ for $k_1=k_2=\cdots=k_n=1$ is called the n - ***dimensional unit cube***, which is denoted by E^n.

For a Boolean vector $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ the set $\alpha(T^n_{k_1k_2\cdots k_n})=\{x\in T^n_{k_1k_2\cdots k_n} / \delta(x)=\alpha\}$ is called α - ***part*** of the torus $T^n_{k_1k_2\cdots k_n}$. It is clear that $T^n_{k_1k_2\cdots k_n}=\bigcup_{\alpha\in E^n}\alpha(T^n_{k_1k_2\cdots k_n})$ and all α-parts of the torus are isomorphic. Notice also that α-parts of $T^n_{k_1k_2\cdots k_n}$ are arranged according to order $\Leftarrow$.

Let $A=\bigcup_{j=-k_i+1}^{k_i}A_i(j)$. We replace each $A_i(j)$ with the standard arrangement in $T_i^n(j)$ of the same cardinality, and call this transformation N_i - ***normalization*** of A with respect to the i - th axis. We denote by $N_i(A)$. the resulting configuration.

It is clear that during N_i - normalization, if some $A_i(j)$ is not the standard arrangement in the corresponding $(n-1)$ - dimensional space $T_i^n(j)$, then instead of some vertices of A we take the same amount of new vertices, precedeing those in the linear ordering $\Leftarrow$ in the $T^n_{k_1k_2\cdots k_n}$. Therefore, if we alternately normalize A with respect to axes 1, 2, ..., n, then after a finite number of steps we obtain ***a stable subset*** A with respect to N_i - normalization, i.e. $N_i(A)=A$ for each i, $1\le i\le n$.

Some properties of the standard arrangement of discrete torus $T^n_{k_1k_2\cdots k_n}$ are proved in [2]; in particular, it is shown that the standard arrangement is stable with respect to N_i - normalization. In this paper we study properties of arbitrary stable subset $A\subseteq T^n_{k_1k_2\cdots k_n}$ with respect to the N_i - normalization.

Observe that in the n-dimensional unit cube the difference between the standard arrangement and stable subsets with respect to the N_i - normalization, - is very small [1].

2. DESCRIPTION OF THE STABLE SUBSETS

Hereafter we shall assume that $n\ge 3$. In this section we give a description of the stable subsets of the discrete torus $T^n_{k_1k_2\cdots k_n}$.

It is proved in [3] that if a subset $A\subseteq T^n_{k_1k_2\cdots k_n}$ is stable with respect to N_i - normalization and $A_n(j_1)=T_n^n(j_1)$ for some $j_1\ge 1$, then $A_n(j)=T_n^n(j)$ for each j, $-j_1+1\le j\le j_1$. Let $j_0\ge 1$ be the smallest number that does not satisfy the condition $A_n(j)=T_n^n(j)$. Then, according to the statement E of the theorem proved in [3], subsets $A_n(j_0)$ and $A_n(-j_0+1)$ can be only of the following types:

- $A_n(-j_0+1)=S_n^n((-j_0+1)e_n,k+1)\bigcup S_{-j_0+1}$, $A_n(j_0)=S_n^n(j_0e_n,k)\bigcup S_{j_0}$, where $\varnothing\ne S_{-j_0+1}\subseteq O_n^n((-j_0+1)e_n,k+2)$, $S_{j_0}\subseteq O_n^n(j_0e_n,k+1)$ and $k+1\le\sum_{i=1}^{n-1}k_i-1$; or
- $A_n(-j_0+1)=S_n^n((-j_0+1)e_n,k)\bigcup S_{-j_0+1}$, $A_n(j_0)=S_n^n(j_0e_n,k)\bigcup S_{j_0}$, where $S_{-j_0+1}\subseteq O_n^n((-j_0+1)e_n,k+1)$, $\varnothing\ne S_{j_0}\subseteq O_n^n(j_0e_n,k+1)$ and $k+1\le\sum_{i=1}^{n-1}k_i-1$.

Hereafter, without loss of generality (for simplicity), we shall assume that $j_0=1$.

One of the following theorems holds.

Theorem 1. If the set $A \subseteq T^n_{k_1 k_2 \cdots k_n}$ is the stable subset of the discrete torus and $A_n(0) = S^n_n(\tilde{0}, k+1) \bigcup S_0$, $A_n(1) = S^n_n(e_n, k) \bigcup S_1$, where $\varnothing \neq S_0 \subseteq O^n_n(\tilde{0}, k+2)$, $S_1 \subseteq O^n_n(e_n, k+1)$, $k+1 \leq \sum_{i=1}^{n-1} k_i - 1$, then $A_n(j) = S^n_n(je_n, k+1-|j|) \bigcup S_j$ for each $j, -k_n+1 \leq j \leq k_n$, where $S_j \subseteq O^n_n(je_n, k+2-|j|)$. Moreover,

1a. if $O^n_n(\tilde{0}, k+2) \bigcap S_0 \neq \varnothing$ only in the first α -part, then

- $O^n_n(je_n, k+2-j) \subseteq A_n(j)$ for each $j, 1 \leq j \leq k_n$, in all α -parts except, perhaps, the last two (when $n=3$ and $k_3 \geq j > k+2-k_1$ it could be that $O^3_3(je_3, k+2-j) \not\subset A_3(j)$ also in the second α -part), and
- $O^n_n(je_n, k+2+j) \bigcap S_j \neq \varnothing$ for each $j, -k_n+1 \leq j \leq 0$, only in the first α -part;

1b. if $O^n_n(\tilde{0}, k+2) \bigcap S_0 \neq \varnothing$ only in the first two α -parts (when $O^n_n(\tilde{0}, k+2) = \{(k_1, k_2, \ldots, k_{n-1}, 0)\}$ we have in mind the first two α -parts of the envelope $O^n_n(-e_n, k+1)$, for which $O^n_n(-e_n, k+1) \bigcap A \neq \varnothing$), then

- $O^n_n(je_n, k+2-j) \subseteq A_n(j)$ for any $j, 1 \leq j \leq k_n$, in all α -parts, except perhaps in the last α -part, and
- $O^n_n(je_n, k+2+j) \bigcap S_j \neq \varnothing$ for each $j, -k_n+1 \leq j \leq 0$, only in the first two α -parts;

1c. if $O^n_n(\tilde{0}, k+2) \bigcap S_0 \neq \varnothing$ at least in the first three α -parts (at $O^n_n(\tilde{0}, k+2) = \{(k_1, k_2, \ldots, k_{n-1}, 0)\}$ we have in mind the first three α -parts of the envelope $O^n_n(-e_n, k+1)$, for which $O^n_n(-e_n, k+1) \bigcap A \neq \varnothing$), then $O^n_n(je_n, k+2-j) \subseteq A_n(j)$ for each $j, 1 \leq j \leq k_n$ (when $n=3$ and $k_2+1 \leq k+2 \leq k_3+k_2-1$ for $j > k_1$ it could be, that $O^3_3(je_3, k+2-j) \not\subset A_3(j)$ in the final fourth α -part);

1d. if $O^n_n(\tilde{0}, k+2) \bigcap S_0 = \varnothing$ in the first α -part and $O^n_n(\tilde{0}, k+2) \bigcap A \neq \varnothing$, then $O^n_n(je_n, k+2-j) \subseteq A_n(j)$ for any $j, 1 \leq j \leq k_n$, or $A = S^n(\tilde{0}, k+1) \bigcup$
$\bigcup \left(\bigcup_{j=1}^{k_n} O^n_n(je_n, k+2-j) \right) \bigcup \{(x_1, x_2, \ldots, x_r, 1, 1, \ldots, 1, 0)\} \setminus$
$\setminus \{(1, 1, \ldots, 1, -k_{r+1}+1, \ldots, -k_{n-1}+1, k_n)\}$, where $x_1 = x_2 = \cdots = x_r = 0$, $k_1 = k_2 = \cdots = k_{r+1} = 1$, $k+2 = n-r-1 = r + \sum_{i=r+1}^{n-1}(k_i - 1) + k_n, r \geq 1$.

Theorem 2. If the set $A \subseteq T^n_{k_1 k_2 \cdots k_n}$ is the stable subset of the discrete torus and $A_n(0) = S^n_n(\tilde{0}, k+1) \bigcup S_0$, $A_n(1) = S^n_n(e_n, k) \bigcup S_1$, where $S_0 \subseteq O^n_n(\tilde{0}, k+1)$, $\varnothing \neq S_1 \subseteq O^n_n(e_n, k+1)$, $k+1 \leq \sum_{i=1}^{n-1} k_i - 1$, then

- ✓ $A_n(j) = S^n_n(je_n, k+1-j) \bigcup S_j$ for each $j, 1 \leq j \leq k_n$, where $S_j \subseteq O^n_n(je_n, k+2-j)$, and
- ✓ $A_n(j) = S^n_n(je_n, k+j) \bigcup S_j$ for each $j, -k_n+1 \leq j \leq 0$, where $S_j \subseteq O^n_n(je_n, k+1+j)$.

Moreover,

2a. if $O^n_n(e_n, k+1) \bigcap S_1 \neq \varnothing$ only in the first α -part, then

- $O^n_n(je_n, k+1+j) \subseteq A_n(j)$ for each $j, -k_n+1 \leq j \leq 0$, in all α -parts except, perhaps, the last two (at $n=3$ for some $j>1$, may be that $O^3_3(-je_3, k+1-j) \not\subset A_3(-j)$ also in the second α -part), and
- $O^n_n(je_n, k+2-j) \bigcap A \neq \varnothing$ for each $j, 1 \leq j \leq k_n$, only in the first α -part;

2b. if $O^n_n(\tilde{0}, k+1) \bigcap S_1 \neq \varnothing$ only in the first two α -parts, then

- $O^n_n(-je_n, k+1-j) \subseteq A_n(-j)$ for each $0 \leq j < k_n$ in all α -parts, except perhaps in the last α -part, and
- $O^n_n(je_n, k+2-j) \bigcap A \neq \varnothing$ only in the first two α -parts for each $j, 0 < j \leq k_n$;

2c. if $O^n_n(e_n, k+1) \bigcap S_1 \neq \varnothing$, at least in the first three α -parts, then $O^n_n(-je_n, k+1-j) \subseteq A_n(-j)$ for each $j, 0 \leq j < k_n$ (when $n=3$ for some $j > k_1 - 1$, it can be, that $O^3_3(-je_3, k+1-j) \not\subset A_3(-j)$ in the final fourth α -part);

2d. if $O^n_n(e_n, k+1) \bigcap S_1 = \varnothing$ in the first α -part and $O^n_n(e_n, k+1) \bigcap A \neq \varnothing$, then $O^n_n(-je_n, k+1-j) \subseteq A_n(-j)$ for any $j, 0 \leq j < k_n$, or $A = S^n(\tilde{0}, k+1) \bigcup$
$\bigcup \{(x_1, x_2, \ldots, x_r, 1, 1, \ldots, 1, 1)\} \setminus \{(1, 1, \ldots, 1, -k_{r+1}+1, \ldots, -k_n+1)\}$, where $x_1 = x_2 = \cdots = x_r = 0$, $k_1 = k_2 = \cdots = k_{r+1} = 1$, $k+1 = n-r-1 = r + \sum_{i=r+1}^{n}(k_i - 1), r \geq 1$.

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