Numerical Algorithms for a Solution of Quasi-Linear Second Order Partial Differential Equation of Mixed Type

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ABSTRACT

In this work the initial problem for one class of differential equations is considered, the main part of which is a non-strictly hyperbolic second-order operator L. When solving the Cauchy problem, in cases of concrete initial conditions there are constructed numerical algorithms for finding the unknown solution.

Keywords

Numerical algorithms, differential equations

In this work the initial problem for one class of differential equations is considered:

$$L(u)=(u_y^2-u_y)u_{xx}-(2u_xu_y+u_y-u_x-1)u_{xy}+ \\ +(u_x^2+u_x)u_{yy}=F(u_x,u_y,y) \tag{1}$$

the main part of which is a non-strictly hyperbolic second-order operator L. Characteristic roots defined by it

$$\lambda_1=-\frac{p+1}{q},\ \lambda_2=\frac{p}{1-q}\ , \tag{2}$$

where $p\equiv u_x$, $q\equiv u_y$ are Monge designations, behave differently with different functions $u\in C^1(R^2)$: with some $u(x,y)$ they may coincide at all points. Then along such functions operator (1) ceases to be hyperbolic and parabolically degenerates. This class of the function is defined by means of the condition

$$p-q+1=0 \tag{3}$$

If the solution of the given equation belongs to this class it will be a parabolic solution. It follows from the structure of the roots (2) that when having parabolic solutions their values not only coincide but they both equal $\lambda_1=\lambda_2=-1$. Accordingly, in such a case characteristic directions coincide with the direction of the family of lines $x+y=c$. If the condition (3) is not fulfilled at all points but only at the determined number of points then the solution is related to the parabolically degenerated hyperbolic class.

We consider the initial problem in concrete case when

$$F=\frac{1}{y}p(p+1)(p-q+1).$$

In this particular case the equation can be fully integrated and its general integral has the following form:

$$f(u+x)+g(u-y)=y^2, \tag{4}$$

with the arbitrary functions $f,g\in C^2(R^1)$.

In the following cases the numerical algorithm is constructed to find a solution of the initial problem.

Case 1. The Cauchy problem is considered when the initial data support is entirely the segment of the straight line x=0, where the following conditions are fulfilled:

$$\begin{gathered} u(0,y)=\varphi(y) \\ a\le y\le b,\ \ a>0 \\ u_x(0,y)=\psi(y). \end{gathered} \tag{5}$$

$\varphi\in C^2[a,b]$, $\psi\in C^1[a,b]$ are the given functions.

It is proved that if the functions φ, ψ are strictly monotonous on the segment [a, b] and the functions φ and $\varphi-y$ correspondingly, have unique reverses **k** and **h**, then there exists the integral of the considered problem and it is represented by the formula

$$\int_{h(u+x)}^{k(u-y)}\frac{2t\psi(t)\varphi'(t)dt}{\varphi'(t)-\psi(t)-1}+[k(u-y)]^2=y^2 \tag{6}$$

On the basis of this representation we can directly obtain implicit equations of characteristics of both families coming from the points $(0,c)$, $a\le c\le b$ of the support. They have a form:

$$\int_{c}^{k(\varphi(c)-x-y)}\frac{2t\psi(t)\varphi'(t)dt}{\varphi'(t)-\psi(t)-1}+[\varphi(c)-x-y]^2=y^2 \tag{7}$$

$$\int_{h(\varphi(c)-c-x-y)}^{c}\frac{2t\psi(t)\varphi'(t)dt}{\varphi'(t)-\psi(t)-1}+c^2=y^2 \tag{8}$$

where we can regard the ordinate **c** as a parameter for both numbers of curves.

For visual demonstration we present in this work the cases of simple initial perturbation and give their numerical realization. In the regular case we prove that the domain of definition of the integral of the Cauchy problem (5) is bounded by the characteristics (7), (8) when the values of the parameter c are $c=a,\ c=b$.

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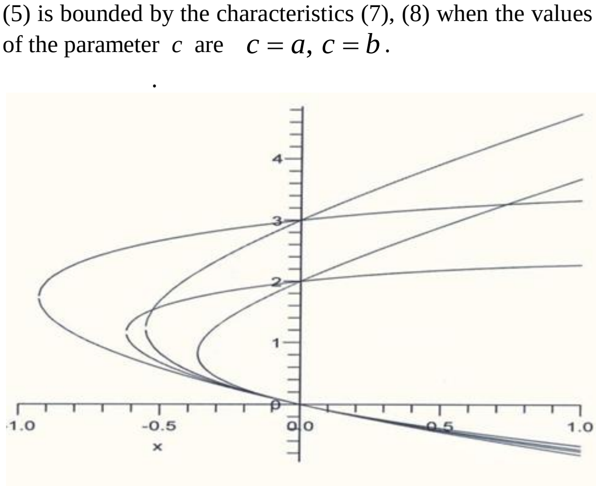

Case 2. If we take the following concrete initial perturbations

$$u|_{x=0}-2 - = y+\sqrt{4y^2+6y}$$

$$u_x|_{x=0}+2 - = \frac{2y+3}{\sqrt{4y^2+6y}}$$

It is possible to construct a numerical algorithm to find the solution for the Initial problem. This solution is written as follows:

$$u=-2-y-2x+\sqrt{2x^2+4y^2+4xy+6x+6y}$$

Here is a finite difference scheme for the problem:

$$\omega_h = \{(\tilde{x}_i^0, \tilde{y}_i^0), \tilde{x}_i^0 = 0, \tilde{y}_i^0 = a + ih, i = 0, \dots, N\},$$

$$\tilde{y}_i^{n+1} - \tilde{y}_i^n = -\frac{\tilde{p}_i^n + 1}{\tilde{q}_i^n}(\tilde{x}_i^{n+1} - \tilde{x}_i^n),$$

$$\tilde{y}_i^{n+1} - \tilde{y}_{i+1}^n = \frac{\tilde{p}_i^n}{1-\tilde{q}_i^n}(\tilde{x}_i^{n+1} - \tilde{x}_{i+1}^n),$$

$$\tilde{p}_i^{n+1} - \tilde{p}_i^n + \frac{\tilde{p}_i^n}{1-\tilde{q}_i^n}(\tilde{q}_i^{n+1} - \tilde{q}_i^n) = 0,$$

$$\tilde{p}_i^{n+1} - \tilde{p}_i^n - \frac{\tilde{p}_i^n + 1}{\tilde{q}_i^n}(\tilde{q}_i^{n+1} - \tilde{q}_i^n) = 0,$$

$$\tilde{u}_i^{n+1} = \frac{1}{2}(\tilde{u}_i^n + \tilde{p}_i^n(\tilde{x}_i^{n+1} - \tilde{x}_i^n) + \tilde{q}_i^n(\tilde{y}_i^{n+1} - \tilde{y}_i^n)) +$$

$$+ \frac{1}{2}(\tilde{u}_{i+1}^n + \tilde{p}_{i+1}^n(\tilde{x}_i^{n+1} - \tilde{x}_{i+1}^n)\tilde{q}_{i+1}^n(\tilde{y}_i^{n+1} - \tilde{y}_{i+1}^n),$$

$$i, n = 0,1,\dots,N-1;$$

$$\tilde{u}_i^0 = u(0, \tilde{y}_i^0), \quad \tilde{p}_i^0 = u_x(0, \tilde{y}_i^0), \quad \tilde{q}_i^0 = u_y(0, \tilde{y}_i^0),$$

$$i = 0,1,\dots,N-1.$$

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