Numerical Solution of Nonlocal Contact Problems for Elliptic Equations

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ABSTRACT
The present work is devoted to the statement and analysis of one nonlocal contact problem for Poisson's equation in two-dimensional domain. For numerical solution the iteration process is constructed, which allows one to reduce the solution of the initial problem to the solution of a sequence of classical Dirichlet problems. The algorithm is suitable for parallel realization. The specific problem is considered as example and solved numerically by using Wolfram Mathematica.

Keywords
Contact problem, nonlocal problem, iteration algorithm, numerical solution, elliptic equation

1. INTRODUCTION
The history of investigation of nonlocal boundary value problems began in the 20th century. The publication of [1-2] gave impetus to numerous studies in this direction.
In the Cannon's paper the nonlocal problem was stated as

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\ 0<x<1,\ 0<t<T,$$

$$u(0,t)=\phi_1(t),\qquad \int_0^1 u(x,t)dx=\phi_2(t),$$

$$u(x,0)=u_0(x),$$

where $\phi_1(.)$, $\phi_2(.)$, $u_0(.)$ are known smooth functions, which satisfy the coordination conditions. This was nonlocal problem, which laid the foundation for the new direction in research of nonlocal boundary problems and methods of their numerical solution.
As for A.Bitsadze, A.Samarskii's problem, it was stated in general form, but the uniqueness of the solution and its resolvability was proved in case of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\qquad -l<x<l,$$

$$0<y<1,\qquad l=const>0,$$

with boundary conditions

$$u(x,0)=\phi_1(x),\qquad u(x,1)=\phi_2(x),\qquad -l\le x\le l,$$

$$u(-l,y)=\phi_3(y),\qquad 0\le y\le 1,$$

$$u(0,y)=u(l,y),\qquad 0\le y\le 1.$$

Here $\phi_i(\ )$, $(i=\overline{1,3})$ are known continuous functions.
The results obtained in these works were generalized and refined from the standpoints of theory and application.
In the last two decades extensive studies of nonlocal initial-boundary and boundary value problems were carried out, general theoretical fundamental principles of analysis were formulated, methods were developed for the numerical solution of problems and the construction of mathematical models of concrete problems in physics, ecology, biology, economics and other areas (see [3-10] – the results of D.Gordeziani, H.Meladze, M.Sapagovas, V.Makarov, G.Berikelashvili, G.Avalishvili, D.Kapanadze, etc.).
In article [11] the boundary problem with nonlocal contact conditions for linear operator equations is stated and investigated in two-dimensional area.

2. STATEMENT OF THE PROBLEM
Let us consider the rectangle domain D in two-dimensional space R^2:

$$D=\{(x_1,x_2)\,|\,0<x_1<a,\quad 0<x_2<b\}$$

with piecewise boundary $\Gamma=\bigcup_{i=1}^{4}\Gamma_i$, where

$$\Gamma_1=\{(x_1,x_2)\,|\,0\le x_1\le a,\quad x_2=0\},$$
$$\Gamma_2=\{(x_1,x_2)\,|\,0\le x_1\le a,\quad x_2=b\},$$
$$\Gamma_3=\{(x_1,x_2)\,|\,x_1=0,\quad 0\le x_2\le b\},$$
$$\Gamma_4=\{(x_1,x_2)\,|\,x_1=a,\quad 0\le x_2\le b\}.$$

Let us consider also the segments:

$$\Gamma_0=\{(x_1,x_2)\,|\,x_1=\xi^0,\quad 0<\xi^0<a,\quad 0\le x_2\le b\},$$
$$\Gamma^-=\{(x_1,x_2)\,|\,x_1=\xi^-,\quad 0<\xi^-<\xi^0,\quad 0\le x_2\le b\},$$
$$\Gamma^+=\{(x_1,x_2)\,|\,x_1=\xi^+,\quad \xi^0<\xi^+<a,\quad 0\le x_2\le b\},$$

which intersect Γ_1 and Γ_2, respectively, in the points (see Fig.1): $A^0(\xi^0,0)$, $B^0(\xi^0,b)$, $A^-(\xi^-,0)$, $B^-(\xi^-,b)$, $A^+(\xi^+,0)$ and $B^+(\xi^+,b)$.

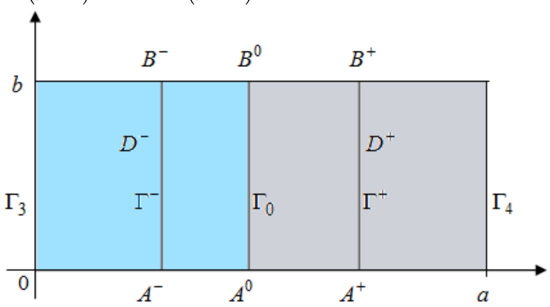


Fig. 1

It is obvious that Γ_0 divides the domain D into two parts (domains) D^- and D^+, where

$$D^-=\{(x_1,x_2)\,|\,0<x_1<\xi^0,\quad 0<x_2<b\},$$
$$D^+=\{(x_1,x_2)\,|\,\xi^0<x_1<a,\quad 0<x_2<b\}.$$

Further, the following notations are needed:

$$\Gamma_1^- = \{(x_1, x_2) \mid 0 \le x_1 \le \xi^0, \quad x_2 = 0\},$$
$$\Gamma_2^- = \{(x_1, x_2) \mid 0 \le x_1 \le \xi^0, \quad x_2 = b\},$$
$$\Gamma_1^+ = \{(x_1, x_2) \mid \xi^0 \le x_1 \le a, \quad x_2 = 0\},$$
$$\Gamma_2^+ = \{(x_1, x_2) \mid \xi^0 \le x_1 \le a, \quad x_2 = b\}.$$

We consider the following problem: find in $\overline{D}$ a continuous function $u(x_1, x_2)$,

$$u(x_1, x_2) = \begin{cases} u^-(x_1, x_2), & if\ (x_1, x_2) \in D^-, \\ u^0(x_1, x_2), & if\ (x_1, x_2) \in \Gamma_0, \\ u^+(x_1, x_2), & if\ (x_1, x_2) \in D^+, \end{cases} \quad (1)$$

$u^-(x_1, x_2) \in C^2(D^-)$, $u^+(x_1, x_2) \in C^2(D^+)$, $u^0(x_1, x_2) \in C(\Gamma_0)$, which satisfies the equations

$$\Delta u^-(x_1, x_2) = f^-(x_1, x_2), \quad if\ (x_1, x_2) \in D^-, \quad (2)$$

$$\Delta u^+(x_1, x_2) = f^+(x_1, x_2), \quad if\ (x_1, x_2) \in D^+, \quad (3)$$

where $f^-(x_1, x_2)$ and $f^+(x_1, x_2)$ are known, sufficiently smooth functions.

The function $u(x_1, x_2)$ satisfies the boundary conditions:

$$u^-(x_1, x_2) = \varphi^-(x_1, x_2), \quad if\ (x_1, x_2) \in \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3, \quad (4)$$

$$u^+(x_1, x_2) = \varphi^+(x_1, x_2), \quad if\ (x_1, x_2) \in \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_4, \quad (5)$$

the nonlocal contact conditions:

$$u^-(\Gamma_0) = u^+(\Gamma_0) = u(\Gamma_0) = = \gamma^+ u^+(\Gamma^+) + \gamma^- u^-(\Gamma^-) + \varphi^0(\Gamma_0), \quad (6)$$

and the coordination conditions

$$u(A^0) = \gamma^+ u^+(A^+) + \gamma^- u^-(A^-) + \varphi^0(A^0), \quad (7)$$

$$u(B^0) = \gamma^+ u^+(B^+) + \gamma^- u^-(B^-) + \varphi^0(B^0), \quad (8)$$

where

$\gamma^- = const \ge 0$, $\gamma^+ = const \ge 0$, $0 < \gamma^- + \gamma^+ \le 1$, $\varphi^0(\cdot)$, $\varphi^-(\cdot)$ and $\varphi^+(\cdot)$ are known sufficiently smooth functions.

We will call the problem (1)-(8) nonlocal contact one since it is generalization of a classical contact problem.

3. UNIQUENESS OF A SOLUTION OF PROBLEM (1)-(8)

The following theorem is true.

Theorem 1. If the regular solution of problem (1)-(8) exists and condition $0 < \gamma^- + \gamma^+ \le 1$ is fulfilled, then the solution is unique.

Proof. Suppose that problem (1)-(8) has two solutions: $v(x_1, x_2)$ and $w(x_1, x_2)$. Then for the function $z(x_1, x_2) = v(x_1, x_2) - w(x_1, x_2)$ we have the following problem

$$\Delta z^- = 0, \quad if\ (x_1, x_2) \in D^-, \quad (9)$$

$$\Delta z^+ = 0, \quad if\ (x_1, x_2) \in D^+, \quad (10)$$

$$z^- = 0, \quad if\ (x_1, x_2) \in \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3,$$
$$z^+ = 0, \quad if\ (x_1, x_2) \in \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_4, \quad (11)$$

$$z(\Gamma_0) = z^-(\Gamma_0) = z^+(\Gamma_0) = \gamma^+ z^+(\Gamma^+) + \gamma^- z^-(\Gamma^-). \quad (12)$$

From equality (12) it follows that

$$\max|z(\Gamma_0)| \le \gamma^+ \max|z^+(\Gamma^+)| + \gamma^- \max|z^-(\Gamma^-)|.$$

Taking into account the condition $0 < \gamma^- + \gamma^+ \le 1$, we obtain

$$\max|z(\Gamma_0)| \le \max|z^+(\Gamma^+)| \quad or$$
$$\max|z(\Gamma_0)| \le \max|z^-(\Gamma^-)|.$$

This means that the function z does not attain a maximum on Γ_0, but attains its maximum on D^- or D^+, that contradicts the maximum principle. So, $z \equiv const$ and taking condition (11) into account, we obtain $z \equiv 0$, i.e. the solution of problem (1)-(8) is unique. ◆

4. EXISTENCE OF A SOLUTION OF PROBLEM (1)-(8)

Let us consider the following iteration process:

$$\Delta(u^-(x_1, x_2))^{(k)} = f^-(x_1, x_2), \quad if\ (x_1, x_2) \in D^-, \quad (13)$$

$$\Delta(u^+(x_1, x_2))^{(k)} = f^+(x_1, x_2), \quad if\ (x_1, x_2) \in D^+, \quad (14)$$

$$(u^-(x_1, x_2))^{(k)} = \varphi^-(x_1, x_2), \quad if\ (x_1, x_2) \in \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3, \quad (15)$$

$$(u^+(x_1, x_2))^{(k)} = \varphi^+(x_1, x_2), \quad if\ (x_1, x_2) \in \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_4, \quad (16)$$

$$u^{(k)}(\Gamma^0) = [u^-(\Gamma^0)]^{(k)} = [u^+(\Gamma^0)]^{(k)} = = [\gamma^+ u^+(\Gamma^+)]^{(k-1)} + [\gamma^- u^-(\Gamma^-)]^{(k-1)} + \varphi^0(\Gamma_0), \quad (17)$$

where

$k = 0,1,2,\ldots$ and $(u^-)^{(-1)}(\Gamma^-) = 0$, $(u^+)^{(-1)}(\Gamma^+) = 0$.

Denote

$$[z^-(x_1, x_2)]^{(k)} = [u^-(x_1, x_2)]^{(k)} - u^-(x_1, x_2), \quad if\quad u^-(x_1, x_2) \in \overline{D}^-,$$
$$[z^+(x_1, x_2)]^{(k)} = [u^+(x_1, x_2)]^{(k)} - u^+(x_1, x_2), \quad if\quad u^+(x_1, x_2) \in \overline{D}^+.$$

Then for the function $z(x_1, x_2)$ we obtain the problem

$$\Delta(z^-(x_1, x_2))^{(k)} = 0, \quad if\ (x_1, x_2) \in D^-, \quad (18)$$

$$\Delta(z^+(x_1, x_2))^{(k)} = 0, \quad if\ (x_1, x_2) \in D^+, \quad (19)$$

$$(z^-(x_1, x_2))^{(k)} = 0, \quad if\ (x_1, x_2) \in \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3, \quad (20)$$

$$(z^+(x_1, x_2))^{(k)} = 0, \quad if\ (x_1, x_2) \in \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_4, \quad (21)$$

$$[z(\Gamma^0)]^{(k)} = [z^-(\Gamma^0)]^{(k)} = [z^+(\Gamma^0)]^{(k)} = [\gamma^+ z^+(\Gamma^+)]^{(k-1)} + [\gamma^- z^-(\Gamma^-)]^{(k-1)}. \quad (22)$$

From equality (22) we have

$$\max_{\Gamma_0} \left| [z(\Gamma_0)]^{(k)} \right| \le \gamma^+ \max_{\Gamma^+} \left| [z^+(\Gamma^+)]^{(k-1)} \right| + + \gamma^- \max_{\Gamma^-} \left| [z^-(\Gamma^-)]^{(k-1)} \right|.$$

If we use Schwarz' lemma, we will get inequalities:

$$\max_{\Gamma^+}\left|\left[z^+\left(\Gamma^+\right)\right]^{(k)}\right| \le q^+ \max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k-1)}\right|, \qquad (23)$$

$$\max_{\Gamma^-}\left|\left[z^-\left(\Gamma^-\right)\right]^{(k)}\right| \le q^- \max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k-1)}\right|, \qquad (24)$$

where

$q^+ = const, \quad 0 < q^+ < 1, \quad q^- = const, \quad 0 < q^- < 1$.

Note, that these constants depend only on geometric properties of domains D^- and D^+.

If we use the nonlocal contact conditions (22) and inequalities (23), (24), then we have

$$\max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k)}\right| \le \gamma^+ q^+ \max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k-1)}\right| + $$
$$+ \gamma^- q^- \max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k-1)}\right|,$$

or

$$\max\left|\left[z(\Gamma_0)\right]^{(k)}\right| \le Q \max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k-1)}\right|, \qquad (25)$$

where

$$Q = \gamma^+ q^+ + \gamma^- q^- .$$

Taking into account the conditions

$$\gamma^+ \ge 0, \ \gamma^- \ge 0, \quad 0 < \gamma^- + \gamma^+ \le 1,$$

we obtain $0 < Q < 1$. This implies that

$$\lim_{k\to\infty} z^{(k)}(\Gamma_0) = 0 .$$

If the solution of problem (1)-(8) exists, then by the maximum principle we obtain

$$\max_{D^-}\left|\left[u^-(x_1,x_2)\right]^{(k)} - u^-(x_1,x_2)\right| = O\left(Q^k\right),$$

$$\max_{D^+}\left|\left[u^+(x_1,x_2)\right]^{(k)} - u^+(x_1,x_2)\right| = O\left(Q^k\right),$$

and, accordingly,

$$\max_{\overline{D}}\left|\left[u(x_1,x_2)\right]^{(k)} - u(x_1,x_2)\right| = O\left(Q^k\right).$$

Thereby we proved the following theorem.

Theorem 2. If the solution of problem (1)-(8) exists, then the iteration process (13)-(17) converges to this solution at the rate of an infinitely decreasing geometric progression.

Remark. By using the iteration algorithm (13)-(17) the solution of a nonclassical contact problem (1)-(8) is reduced to the solution of a sequence of classical Dirichlet problems.

Let us now prove the existence of a regular solution of problem (1)-(8). We introduce the notation

$$\varepsilon^{(k)}(x_1,x_2) = u^{(k)}(x_1,x_2) - u^{(k-1)}(x_1,x_2) .$$

Then for the function $\varepsilon^{(k)}$ we obtain the problem

$$\Delta\left[\varepsilon^-(x_1,x_2)\right]^{(k)} = 0, \quad if \quad (x_1,x_2)\in D^-, \qquad (26)$$

$$\Delta\left[\varepsilon^+(x_1,x_2)\right]^{(k)} = 0, \quad if \quad (x_1,x_2)\in D^+, \qquad (27)$$

$$\left[\varepsilon^-(x_1,x_2)\right]^{(k)} = 0, \quad if \ (x_1,x_2)\in \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3, \qquad (28)$$

$$\left[\varepsilon^+(x_1,x_2)\right]^{(k)} = 0, \quad if \ (x_1,x_2)\in\in \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_4, \qquad (29)$$

$$\left[\varepsilon\left(\Gamma^0\right)\right]^{(k)} = \left[\varepsilon^+\left(\Gamma^0\right)\right]^{(k)} = \left[\varepsilon^-\left(\Gamma^0\right)\right]^{(k)} =$$
$$= \gamma^+\left[\varepsilon^+\left(\Gamma^+\right)\right]^{(k-1)} + \gamma^-\left[\varepsilon^-\left(\Gamma^-\right)\right]^{(k-1)}, \qquad (30)$$

where $k = 0,1,2,\ldots$ and

$$\left[\varepsilon^-\left(\Gamma^-\right)\right]^{(-1)} = 0, \ \left[\varepsilon^+\left(\Gamma^+\right)\right]^{(-1)} = 0 .$$

Then, analogously to (25), we obtain the estimate

$$\max_{\Gamma^0}\left|\left[\varepsilon\left(\Gamma^0\right)\right]^{(k)}\right| \le Q \max_{\Gamma^0}\left|\left[\varepsilon\left(\Gamma^0\right)\right]^{(k-1)}\right|, \quad 0 < Q < 1,$$

or

$$\max_{\Gamma^0}\left|u^{(k)}\left(\Gamma^0\right) - u^{(k-1)}\left(\Gamma^0\right)\right| \le$$
$$\le Q \max_{\Gamma^0}\left|u^{(k-1)}\left(\Gamma^0\right) - u^{(k-2)}\left(\Gamma^0\right)\right|, \quad 0 < Q < 1.$$

This means that the sequence $\left\{u^{(k)}(x_1,x_2)\right\}$ converges uniformly on Γ_0. Then the functions $\left[u^+(x_1,x_2)\right]^{(k)}$ and $\left[u^-(x_1,x_2)\right]^{(k)}$ converge to the functions $u^+(x_1,x_2)$ and $u^-(x_1,x_2)$ on the base of Harnak's first theorem [12] for the domains D^- and D^+.

From this we conclude that the limit function is the regular solution of problem (1)-(8):

$$\lim_{k\to\infty} u^{(k)}(x_1,x_2) = u(x_1,x_2) .$$

We have thereby proved that by using the iteration algorithm the solution of a nonclassical contact problem is reduced to the solution of a sequence of classical Dirichlet problems.

Note, that the algorithm can be used for parallel realization.

5. EXAMPLE

Let us consider the area

$$D = \left\{(x_1,x_2) \mid 0 < x_1 < 1, \quad 0 < x_2 < 1\right\}:$$

We consider the following problem: find in $\overline{D}$ a continuous function $u(x_1,x_2)$,

$$u(x_1,x_2) = \begin{cases} u^-(x_1,x_2), & if \ (x_1,x_2)\in D^-, \\ u^0(x_1,x_2), & if \ (x_1,x_2)\in \Gamma_0, \\ u^+(x_1,x_2), & if \ (x_1,x_2)\in D^+, \end{cases} \qquad (31)$$

which satisfies the equations

$$\Delta u^-(x_1,x_2) = 2x_1^2(x_1-1)(3x_2-1) +$$
$$+ 2x_2^2(3x_1-1)(x_2-1), \quad if \ (x_1,x_2)\in D^-, \qquad (32)$$

$$\Delta u^+(x_1,x_2) = 2x_1x_2^2(x_2-1) +$$
$$+ \frac{2}{3}x_1\left(x_1^2-1\right)(3x_2-1), \quad if \ (x_1,x_2)\in D^+. \qquad (33)$$

The function $u(x_1,x_2)$ also satisfies the boundary conditions

$$u^-(x_1,x_2) = 0, \quad if \ (x_1,x_2)\in \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3, \qquad (34)$$

$$u^+(x_1,x_2) = 0, \quad if \ (x_1,x_2)\in \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_4, \qquad (35)$$

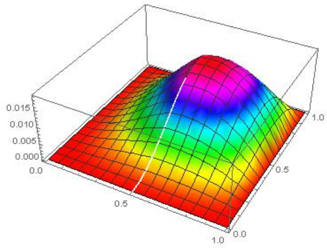
the nonlocal contact conditions

$$u^-\left(\frac{1}{2},x_2\right) = u^+\left(\frac{1}{2},x_2\right) = u(\Gamma_0) = \frac{1}{4}u^-\left(\frac{1}{4},x_2\right) +$$
$$+ \frac{1}{4}u^+\left(\frac{3}{4},x_2\right) - \frac{11}{128}x_2^2(x_2-1) \qquad (36)$$

and the coordination conditions are fulfilled.

The exact solution of this problem is

$$u(x_1,x_2)=\begin{cases} x_1^2 x_2^2 (x_1-1)(x_2-1), & if\ (x_1,x_2)\in D^-, \\ -\frac{1}{8}x_2^2(x_2-1), & if\ (x_1,x_2)\in \Gamma_0, \\ \frac{1}{3}x_1x_2^2(x_1^2-1)(x_2-1), & if\ (x_1,x_2)\in D^+, \end{cases}$$



Let us consider the following iteration process:

$$\Delta\left(u^-(x_1,x_2)\right)^{(k)} = 2x_1^2(x_1-1)(3x_2-1)+ \\ +2x_2^2(3x_1-1)(x_2-1), \quad if\ (x_1,x_2)\in D^-, \tag{37}$$

$$\Delta\left(u^+(x_1,x_2)\right)^{(k)} = 2x_1x_2^2(x_2-1)+ \\ +\frac{2}{3}x_1(x_1^2-1)(3x_2-1), \quad if\ (x_1,x_2)\in D^+, \tag{38}$$

$$\left(u^-(x_1,x_2)\right)^{(k)}=0, \quad if\ (x_1,x_2)\in \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3, \tag{39}$$

$$\left(u^+(x_1,x_2)\right)^{(k)}=0, \quad if\ (x_1,x_2)\in \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_4, \tag{40}$$

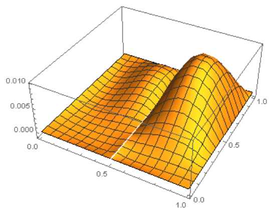
$$u^{(k)}\left(\Gamma^0\right)=\left[u^-\left(\Gamma^0\right)\right]^{(k)}=\left[u^+\left(\Gamma^0\right)\right]^{(k)}=\left[\frac{3}{4}u^+\left(\Gamma^+\right)\right]^{(k-1)}+ \\ +\left[\frac{1}{4}u^-\left(\Gamma^-\right)\right]^{(k-1)}-\frac{11}{128}x_2^2(x_2-1), \tag{41}$$

where

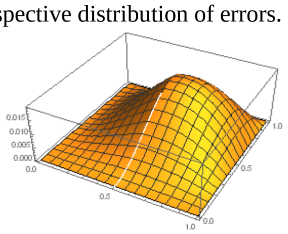
$k=0,1,2,\ldots$ and $\left(u^-\right)^{(-1)}\left(\Gamma^-\right)=0$, $\left(u^+\right)^{(-1)}\left(\Gamma^+\right)=0$.

Initial value for $u^{(k)}\left(\Gamma^0\right)$ is equal to 0.

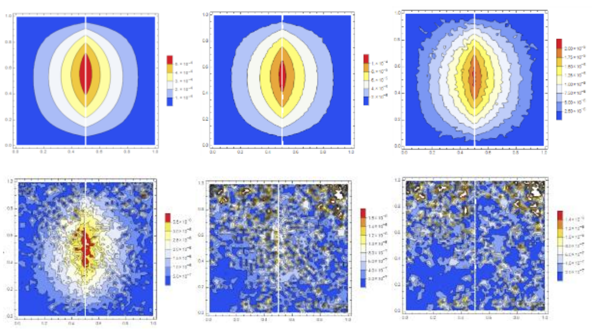
Below one can see the figures of approximate solution for k=1 and k=6 iterations and respective distribution of errors.



$(u(x_1,x_2))^{(k)}$, k=1,



$(u(x_1,x_2))^{(k)}$, k=6



$k=1,\quad 2.\cdot10^{-3}-0.018$, $k=2,\quad 5.\cdot10^{-4}-3.\cdot10^{-3}$, $k=3,\quad 1.\cdot10^{-4}-5.\cdot10^{-4}$, $k=4,\quad 2.\cdot10^{-5}-1.\cdot10^{-4}$, $k=5,\quad 2.5\cdot10^{-6}-2.\cdot10^{-5}$, $k=6,\quad 5.\cdot10^{-7}-3.5\cdot10^{-6}$.

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