Antipodal Sequences and Matrices

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ABSTRACT

In this paper, we investigate antipodal paraunitary (APU) matrices, which is the special case of paraunitary (PU) matrices. APU matrices are PU matrices whose elements are +1 and -1 only. We consider new construction methods of APU matrices based on Agayan-Sarukhanyan [4] and Craigen-Seberry-Zhang [15] multiplicative methods.

Keywords

Antipodal (AP) matrix, antipodal paraunitary (APU) matrix, AP sequence, z-transform, Hadamard matrix, cross-orthogonal sequences, Kronecker product, Tseng's interleaving and concatenating methods, generalized AS method, generalized Craigen-Seberry-Zhang (CSZ) method

1. INTRODUCTION

First, we introduce some definitions and notations from [1].

Antipodal Sequences and Polynomials: A causal sequence $\{a(n)\}_{n=0}^{N-1}$ of length *N* is called an ***antipodal*** (AP) sequence if $a(n)=\pm 1$ for $0 \le n \le N-1$ and 0, otherwise. The z-transform $A(z)=\sum_n a(n)z^{-n}$ of an AP sequence $\{a(n)\}_{n=0}^{N-1}$ will be called an AP polynomial, and (-1,+1)-matrix is called AP matrix. A polynomial matrix $A(z)=\sum_{n=0}^{N-1} A_n z^{-n}$ is AP if A_n are AP matrices.

Examples of antipodal (AP) sequences and polynomials:
AP sequence: $a(n)=\{+1,+1,-1,-1,+1,-1,-1,+1\}$;

Z-transform of AP sequence $a(n)$**:**

$$A(z)=1+z^{-1}-z^{-2}-z^{-3}+z^{-4}-z^{-5}-z^{-6}+z^{-7};$$

AP matrix: any (-1,+1)-matrix

$$A_0=\begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix},\quad A_1=\begin{pmatrix} + & - & - \\ - & + & - \\ - & - & + \end{pmatrix},\quad A_2=\begin{pmatrix} + & - & - \\ - & + & - \\ - & - & + \end{pmatrix},$$

Polynomial matrix:

$$A(z)=\begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix}+\begin{pmatrix} + & - & - \\ - & + & - \\ - & - & + \end{pmatrix}z^{-1}+\begin{pmatrix} + & - & - \\ - & + & - \\ - & - & + \end{pmatrix}z^{-2}.$$

Order, Length, and Degree: If $A_0, A_{N-1} \ne 0$ then the ***order*** and ***length*** of $A(z)$ are, respectively, *N*-1 and *N*. For example, the order of $A(z)=A_0+A_1z^{-1}$ is equal to 1, while its ***degree*** is equal to the rank of A_1 **.**

The order and length of $A(z)$ **are, respectively, 2 and 3.**

$$A_0+A_1z^{-1}=\begin{pmatrix} 1+z^{-1} & 1-z^{-1} & 1-z^{-1} \\ 1-z^{-1} & 1+z^{-1} & 1-z^{-1} \\ 1-z^{-1} & 1-z^{-1} & 1+z^{-1} \end{pmatrix}$$

Tilde Notation*:** The ***tilde $A(z)$ is defined as $\tilde{A}(z)=A^H(1/z^*)$. For a polynomial matrix $A(z)$ with real coefficients we have $\tilde{A}(z)=A^T(z^{-1})$.

The Tilde of $A(z)=A_0+A_1z^{-1}$ has the form

$$\tilde{A}(z)=\begin{pmatrix} 1+z & 1-z & 1-z \\ 1-z & 1+z & 1-z \\ 1-z & 1-z & 1+z \end{pmatrix}$$

Cross Correlation: The *k*-th cross correlation coefficient of two sequences $a(n)$ and $b(n)$ is given by

$$r_{ab}(k)=\sum_n a(n)b^*(n-k).$$

The z-transform of the $r_{ab}(k)$ coefficients can presented as

$$R_{ab}(z)=\sum_k r_{ab}(k)z^{-k}=A(z)\tilde{B}(z).$$

Kronecker Product of Matrices: Let $A(z)=\{A_{i,j}(z)\}_{i,j=0}^{M-1}$ and $B(z)=\{B_{i,j}(z)\}_{i,j=0}^{N-1}$ are square polynomial matrices with dimensions M and N, and orders N_a and N_b, respectively. The *Kronecker product* of these matrices $C(z)=A(z)\otimes B(z)$ is the polynomial matrix of dimension MN and order N_a+N_b defined as

$$C(z)=\begin{pmatrix} A_{0,0}(z)B(z) & A_{0,1}(z)B(z) & \cdots & A_{0,M-1}(z)B(z) \\ A_{1,0}(z)B(z) & A_{1,1}(z)B(z) & \cdots & A_{1,M-1}(z)B(z) \\ \vdots & \vdots & \ddots & \vdots \\ A_{M-1,0}(z)B(z) & A_{M-1,1}(z)B(z) & \cdots & A_{M-1,M-1}(z)B(z) \end{pmatrix}.$$

When we have constant matrices independent of z, then the above definition reduces to the conventional Kronecker product.

Paraunitary and Normalized Paraunitary Matrices: An $N\times N$ polynomial matrix $A(z)$ is said to be paraunitary (PU), if for some nonzero constant c we have

$$\tilde{A}(z)A(z)=|c|^2 I_N \qquad (1)$$

When $|c|=1$, we say that $A(z)$ is a normalized PU matrix. Therefore, $A(e^{j\omega})$ is unitary for all frequency ω. Note that the normalized PU matrices has the energy-preservation property [1]. That is, if the input $x(n)$ of a normalized PU matrix is a vector sequence of finite energy, the output vector $y(n)$ satisfies

$$\sum_n y^H(n)y(n) = \sum_n x^H(n)x(n). \qquad (2)$$

One can show that given two any PU matrices $A(z)$ and $B(z)$, their Kronecker product $A(z)\otimes B(z)$ is also PU.

***Hadamard Matrices*:** $M\times M$ Hadamard matrix H_M, is a constant AP matrix (independent of z) that satisfies

$$H_M^T H_M = MI_M.$$

Note that the necessary condition for the existence of a Hadamard matrix of order M>2 is $M\equiv 0(\bmod 4)$ [2]. Whether this is also a sufficient condition is still not known. Most of Hadamard matrices have been successfully constructed in [3, 4].

Antipodal PU Matrices: If a PU matrix $A(z)$ is also antipodal, then it is called an *antipodal paraunitary* (APU) matrix. We can verify that if an $M\times M$ matrix

$$A(z) = \sum_{n=0}^{N-1} A_n z^{-n} \qquad (3)$$

is APU matrix, then it satisfies

$$\tilde{A}(z)A(z) = A(z)\tilde{A}(z) = MNI_M,$$

and its inverse $\tilde{A}(z)$ is APU.

2. CROSS-ORTHOGONAL SEQUENCES AND APU MATRICES

In this section, we define three types of sequences, which are widely used in communications. We also describe the connection between APU matrices and these sequences.

A set of M antipodal sequences $a_i(n),\ i=\overline{0,M-1}$ of length N, is said to be ***complementary*** [5] if

$$\sum_{i=0}^{M-1} r_{a_i a_i}(k) = MN\delta(k). \qquad (4)$$

For M=2, the complementary sequences are also known as the Golay codes [6].

An antipodal sequence $b(n)$ of length N is called an ***M-shift orthogonal sequence*** [7] if $r_{bb}(lM) = N\delta(l)$. One can verify that $a_i(n) = b(Mn+i)$ are complementary if and only if $b(n)$ is M-shift orthogonal.

A set of antipodal sequences $b_i(n)$ is called a set of ***M-shift cross-orthogonal*** sequences [7] if each $b_i(n)$ sequence is M-shift orthogonal sequences and any pair of distinct sequences $b_i(n)$ and $b_j(n)$ satisfies $r_{b_i b_j}(lM) = 0$ for all integers *l*. These sequences have been extensively studied in [5-8], and they have found many applications in communications [7, 9, 10].

APU Matrices and M-Shift Orthogonal Sequences: From (2) we see that if an $M\times M$ matrix $E(z)$ of length N is APU, then its entries $E_{ki}(z)$ satisfy the following equations

$$\sum_{i=0}^{M-1} E_{ki}(z)\tilde{E}_{ki}(z) = MN, \qquad k=\overline{0,M-1}.$$

Note that this expression is the z-domain formulation of (4). Thus, the M sequences $e_{ki}(n)$ for $i=\overline{0,M-1}$, on the k-th row of an APU matrix $E(z)$ are complementary $(k=\overline{0,M-1})$. Moreover, if we use $E(z)$ as the polyphase matrix of the analysis filterbank [1, 11], then the analysis impulse responses $h_k(n)$ satisfy

$$\sum_n h_j(n)h_k(n+iM) = MN\delta(i)\delta(k-j),\ \ k=\overline{0,M-1}.$$

This means that the impulse response $h_k(n)$ is an M-shift orthogonal sequence, and any two analysis filters form a pair of M-shift cross-orthogonal sequences. Therefore, the study and construction of a set of M-shift cross-orthogonal sequences are identical to those of the APU matrices.

3. CONSTRUCTION METHODS OF APU MATRICES IN Z-DOMAIN

In this section, we present a simple and useful result for the understanding of the followed construction methods. Let's consider two AP polynomials

$$A(z) = \sum_{n=0}^{N-1} a_n z^{-n}, \qquad B(z) = \sum_{n=0}^{M-1} b_n z^{-n}$$

Then, the following four polynomials are AP polynomials

$$A(z)B(z^N), \qquad A(z^M)B(z), \qquad (5)$$
$$A(z)+z^{-N}B(z), \qquad z^{-M}A(z)+B(z).$$

The lengths of the first two and the last two polynomials are NM and $N+M$.

2×2 APU Matrices: APU matrices of size 2×2 are closely related Golay sequences [6, 8]. Recall that a pair of AP sequences $a(n)$ and $b(n)$ of length N are complementary sequences if they satisfy

$$A(z)\tilde{A}(z) + B(z)\tilde{B}(z) = 2N. \qquad (6)$$

Consider the following matrix

$$E(z) = \begin{pmatrix} A(z) & B(z) \\ -z^{-N+1}\tilde{B}(z) & z^{-N+1}\tilde{A}(z) \end{pmatrix}.$$

Using condition (6), it is straightforward to verify that $\tilde{E}(z)E(z) = 2NI$ and $E(z)$ is an APU matrix. Hence, every pair of Golay sequences generates 2×2 APU matrix. Golay sequences with length $N = 2^a 10^b 26^c$ (where a, b, c are integers) have been successfully constructed in [6]. Note that no Golay sequence of other lengths.

Another construction method for 2×2 APU matrices with length equal to a power of two were independently obtained by Shapiro [12] and Rudin [13]. Their construction method is defined recursively as: let $E_0(z) = H_2$ be the 2×2 Hadamard matrix. For $i\ge 1$, let $E_i(z)$ be recursively defined as

$$E_i(z) = H_2 D\left(z^{2^{i-1}}\right)E_{i-1}(z), \qquad (7)$$

where $D(z)$ is the diagonal matrix $D(z) = diag\{1, z^{-1}\}$. As H_2, $D\left(z^{2^{i-1}}\right)$, and $E_{i-1}(z)$ are PU matrices, then their product $E_i(z)$ also is APU matrices of lengths 2^i.

$M\times M$ *APU Matrices:* The idea of Golay sequences were generalized to complementary sequences in [5]. Below we present four known methods of generated APU matrices from the APU matrices of smaller dimensions.

Note that, using the 2×2 APU matrix $E_0(z)$ of length N in [1,5] were recursively generated two different sets of APU matrices.

Tseng's Interleaving Method [5]: For $i \geq 1$, $E_i(z)$ matrix can be defined as

$$E_i(z) = \begin{pmatrix} E_{i-1}(z^2) + z^{-1}E_{i-1}(z^2) & -E_{i-1}(z^2) + z^{-1}E_{i-1}(z^2) \\ -E_{i-1}(z^2) + z^{-1}E_{i-1}(z^2) & E_{i-1}(z^2) + z^{-1}E_{i-1}(z^2) \end{pmatrix}.$$

Tseng's Concatenating Method [5]: For $i \geq 1$, $E_i(z)$ matrix is defined as

$$E_i(z) = \begin{pmatrix} E_{i-1}(z) + z^{-2^{i-1}N}E_{i-1}(z) & -E_{i-1}(z) + z^{-2^{i-1}N}E_{i-1}(z) \\ -E_{i-1}(z) + z^{-2^{i-1}N}E_{i-1}(z) & E_{i-1}(z) + z^{-2^{i-1}N}E_{i-1}(z) \end{pmatrix}.$$

We see that the direct multiplication above matrices gives $\tilde{E}_i(z)E_i(z) = 4^i 2N I_{2^{i+1}}$. Therefore, above defined two matrices are PU matrices. Moreover, using (5) we see that $E_i(z)$ are AP of length $2^i N$. Hence, the interleaving and concatenating methods generate APU matrices.

Tseng's Kronecker Product Method [5]: Let $E_i(z)$ be an $M_1 \times M_1$ APU matrix of length *N*. Let H_{M_2} is Hadamad matrix of order M_2. Consider the matrix $H_{M_2} \otimes E_i(z)$ of order $M_1 M_2$. As the Kronecker product of two PU matrices is also PU, and then the matrix defined above is an APU matrix of length *N*.

4. NEW CONSTRUCTION METHODS FOR APU MATRICES

In this section, we present three new construction methods for APU matrices. The first two methods are simple generalizations of Tseng's Kronecker product method and the Agayan–Sarukhanyan theorem [3, 4]. The last construction method based on the butterfly structure.

Generalized Kronecker Product Method: Let $A(z)$ and $B(z)$ are APU matrices of dimensions M_a and M_b, and of lengths N_a and N_b, respectively. Consider the following two matrices:

$$A(z^{N_b}) \otimes B(z), \quad A(z) \otimes B(z^{N_a}).$$

The antipodal property of these matrices follows directly from (5). Hence, the above two formulas generate two APU matrices with length $N_a N_b$ and dimension $M_a M_b$.

Generalized Agayan–Sarukhanyan (AS) Method: Using the multiplication theorem of Agayan–Sarukhanyan (AS) [3, 4, 14], it was shown that given two Hadamard matrices of dimensions M_a and M_b, one can construct a Hadamard matrix of dimension $M_a M_b / 2$. Below we show that the result can be generalized to the case of APU matrices (see, [1]). Let $A(z)$ and $B(z)$ be APU matrices of dimensions M_a and M_b, respectively. Suppose that their lengths are N_a and N_b, respectively. Consider the following representations of $A(z)$ and $B(z)$ matrices:

$$A(z) = \begin{pmatrix} A_{00}(z) & A_{01}(z) \\ A_{10}(z) & A_{11}(z) \end{pmatrix}, \quad B(z) = \begin{pmatrix} B_{00}(z) & B_{01}(z) \\ B_{10}(z) & B_{11}(z) \end{pmatrix}, \qquad (8)$$

where $A_{ij}(z)$ and $B_{ij}(z)$ are $M_a/2 \times M_a/2$ and $M_b/2 \times M_b/2$ matrices, respectively. This partition is always possible as M_a and M_b are even. Form the following $(M_a M_b)/2 \times (M_a M_b)/2$ matrix with length $N_a N_b$ as:

$$C(z) = \begin{pmatrix} C_{00}(z) & C_{01}(z) \\ C_{10}(z) & C_{11}(z) \end{pmatrix}, \qquad (9)$$

where the sub-matrices are given by

$$C_{00}(z) = \frac{1}{2}[A_{00}(z^{N_b}) + A_{01}(z^{N_b})] \otimes B_{00}(z) + \frac{1}{2}[A_{00}(z^{N_b}) - A_{01}(z^{N_b})] \otimes B_{10}(z),$$
$$C_{01}(z) = \frac{1}{2}[A_{00}(z^{N_b}) + A_{01}(z^{N_b})] \otimes B_{01}(z) + \frac{1}{2}[A_{00}(z^{N_b}) - A_{01}(z^{N_b})] \otimes B_{11}(z),$$
$$C_{10}(z) = \frac{1}{2}[A_{10}(z^{N_b}) + A_{11}(z^{N_b})] \otimes B_{00}(z) + \frac{1}{2}[A_{10}(z^{N_b}) - A_{11}(z^{N_b})] \otimes B_{10}(z),$$
$$C_{11}(z) = \frac{1}{2}[A_{10}(z^{N_b}) + A_{11}(z^{N_b})] \otimes B_{01}(z) + \frac{1}{2}[A_{10}(z^{N_b}) - A_{11}(z^{N_b})] \otimes B_{11}(z).$$

The matrix $C(z)$, formed in such a manner, is called the AS multiplication of $A(z^{N_b})$ and $B(z)$, which are denoted as [1]

$$C(z) = A(z^{N_b}) \otimes_{AS} B(z). \qquad (10)$$

Below we prove that the matrix $C(z)$ is an APU matrix. Define the two matrices $P_i(z) = [A_{i,0}(z) + A_{i,1}(z)]/2$ and $Q_i(z) = [A_{i,0}(z) - A_{i,1}(z)]/2$. Let $P_i(z) = \sum_n p_i(n) z^{-n}$ and $Q_i(z) = \sum_n q_i(n) z^{-n}$ then, one can verify that $p_i(n)$ and $q_i(n)$ consist of either ± 1 or zero. Moreover, if $[p_i(n)]_{k,l} = \pm 1$, then $[q_i(n)]_{k,l} = 0$ and vice versa. Using this result, one can see from (10) that $C(z)$ is antipodal. To show that $C(z)$ is PU, we must to prove that $\tilde{C}(z)C(z) = \alpha I_{M_a M_b/2}$ for some positive constant α. Now, we will show that

$$\tilde{C}_{00}(z)C_{00}(z) + \tilde{C}_{10}(z)C_{10}(z) = \alpha I,$$

where *I* is an identity matrix of order $M_a M_b / 4$. The proof for other terms is very similar. For notational complexity, we will drop the dependency on z. Using the fact that

$$(A(z) \otimes B(z))^{tilda} = \tilde{A}(z) \otimes \tilde{B}(z),$$

we can write

$$\begin{aligned} 4(\tilde{C}_{00}C_{00} + \tilde{C}_{10}C_{10}) \\ = [(\tilde{A}_{00} + A_{01}) \otimes \tilde{B}_{00} + (\tilde{A}_{00} - A_{01}) \otimes \tilde{B}_{10}] \\ \cdot [(A_{00} + A_{01}) \otimes B_{00} + (A_{00} - A_{01}) \otimes B_{10}] \\ + [(\tilde{A}_{10} + A_{11}) \otimes \tilde{B}_{00} + (\tilde{A}_{10} - A_{11}) \otimes \tilde{B}_{10}] \\ \cdot [(A_{10} + A_{11}) \otimes B_{00} + (A_{10} - A_{11}) \otimes B_{10}] \end{aligned}$$

Using the product rule (1), we can obtain

$$4(\tilde{C}_{00}C_{00}+\tilde{C}_{10}C_{10})=$$
$$=[(\tilde{A}_{00}+\tilde{A}_{01})(A_{00}+A_{01})+(\tilde{A}_{10}+\tilde{A}_{11})(A_{10}+A_{11})]\otimes\tilde{B}_{00}B_{00}$$
$$+[(\tilde{A}_{00}-\tilde{A}_{01})(A_{00}-A_{01})+(\tilde{A}_{10}-\tilde{A}_{11})(A_{10}-A_{11})]\otimes\tilde{B}_{10}B_{10}$$
$$+[(\tilde{A}_{00}+\tilde{A}_{01})(A_{00}-A_{01})+(\tilde{A}_{10}+\tilde{A}_{11})(A_{10}-A_{11})]\otimes B_{00}\tilde{B}_{00}$$
$$+[(\tilde{A}_{00}-\tilde{A}_{01})(A_{00}+A_{01})+(\tilde{A}_{10}-\tilde{A}_{11})(A_{10}+A_{11})]\otimes B_{10}\tilde{B}_{00}.$$

Using the fact that $\tilde{A}A=\alpha_a I_{M_a}$, where $\alpha_a>0$, we can simplify the above equation as

$$4(\tilde{C}_{00}C_{00}+\tilde{C}_{10}C_{10})=2\alpha_a I_{M_a}\otimes\tilde{B}_{00}B_{00}+2\alpha_a I_{M_a}\otimes\tilde{B}_{10}B_{10}$$

Because $\tilde{B}B=\alpha_b I_{M_b}$, we can write

$\tilde{C}_{00}C_{00}+\tilde{C}_{10}C_{10}=\alpha_a\alpha_b I_{M_aM_b/4}$, which proves the result.

Modified Wornell Method [9]: Let H_m is a Hadamard matrix of order *m*, and we have diagonal matrix $D_m(z)=diag\{1,z^{-1},z^{-2},\ldots,z^{-m+1}\}$. Consider the following APU matrix.

$$E_{i,m}(z)=H_m D_m(z^{m^{i-1}})E_{i-1,m}(z),\quad E_{0,m}(z)=H_m,\quad (11)$$
$i\geq 1$.

We can verify that the formula (11) is generate the APU matrices of *m*. Give same examples.

1. $i=1,\ m=2$:

$$E_{1,2}(z)=\begin{pmatrix}1+z^{-1} & 1-z^{-1}\\ 1-z^{-1} & 1+z^{-1}\end{pmatrix}=\begin{pmatrix}+ & +\\ + & +\end{pmatrix}+\begin{pmatrix}+ & -\\ - & +\end{pmatrix}z^{-1};$$

2. $i=2,\ m=2$:

$$E_{2,2}(z)=\begin{pmatrix}+ & +\\ + & +\end{pmatrix}+\begin{pmatrix}+ & -\\ + & -\end{pmatrix}z^{-1}+\begin{pmatrix}+ & +\\ - & -\end{pmatrix}z^{-2}+\begin{pmatrix}- & +\\ + & -\end{pmatrix}z^{-3};$$

3. $i=1,\ m=4$:

$$E_{1,4}(z)=\begin{pmatrix}+ & + & + & +\\ + & + & + & +\\ + & + & + & +\\ + & + & + & +\end{pmatrix}+\begin{pmatrix}+ & - & + & -\\ - & + & - & +\\ + & - & + & -\\ - & + & - & +\end{pmatrix}z^{-1}+$$
$$\begin{pmatrix}+ & + & - & -\\ + & + & - & -\\ - & - & + & +\\ - & - & + & +\end{pmatrix}z^{-2}+\begin{pmatrix}+ & - & - & +\\ - & + & + & -\\ - & + & + & -\\ + & - & - & +\end{pmatrix}z^{-3};$$

Generalized Craigen-Seberry-Zhang (CSZ) Method:
Using the CSZ result for construction of Hadamard matrix of order (see, [4, 15]) from four Hadamard matrices of orders m, n, p, and q, can be generalized this result for case of APU matrices.

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