Gossiping Properties of the Edge-Permuted Knödel Graphs

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ABSTRACT

In this paper we consider the gossiping process implemented on several modifications of Knödel graphs. We show the ability of Knödel graphs to remain good network topology for gossiping even in case of cyclic permutation of the weights of its edges. We show that the modified graphs are still able to gossip and not isomorphic to Knödel graphs for any even value of n. The results obtained in this paper makes it possible to construct edge-disjoint paths between any pairs of vertices in the Knödel graph.

Keywords

Graphs, Networks, Telephone problem, Gossip problem, Knödel graphs

1. INTRODUCTION

Gossiping is one of the fundamental problems of information dissemination in communication networks. The gossip problem (also known as a telephone problem) is attributed to A. Boyd (see e.g., [4] for review), although to the best knowledge of the reviewers, it was first formulated by R. Chesters and S. Silverman (Univ. of Witwatersrand, unpublished, 1970). Consider a set of n persons (nodes), each of which initially knows some unique piece of information that is unknown to the others, and they can make a sequence of telephone calls to spread the information. During a call between the given two nodes, they exchange the whole information known to them at that moment. The problem is to find a sequence of calls with a minimum length (minimal gossip scheme), by which all the nodes will know all pieces of the information (complete gossiping). It has been shown in numerous works [1]–[3] that the minimal number of calls is $2n-4$ when $n \geq 4$ and 1, 3 for $n = 2, 3$, respectively. Since then, many variations of gossip problem have been introduced and investigated (see e.g., [4], [5]).

Another variation of the Gossip problem can be formulated by considering the minimum amount of time required to complete gossiping among n persons, where the calls between the non-overlapping pairs of nodes can take place simultaneously, and each call requires one unit of time ([10]).

Obviously, the gossip problem can be easily modeled as a graph whose vertices represent people in gossip scheme, and edges represent calls between people (each edge has weight which represents the moment when communication took place). Thus, the graph is called a complete gossip graph if there are ascending paths between all the pairs of vertices of this graph. A path between two vertices is ascending if each following edge has higher weight than the previous one (when moving from the first edge to the second).

Knödel graphs are one of the 3 well-known network topologies for gossiping/broadcasting (alongside with hypercube and recursive circulant graphs).

Definition 1. *The Knödel graph on $n \geq 2$ vertices (n even) and of maximum degree $\Delta \geq 1$ is denoted by $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the pairs (i,j) with $i = 1,2$ and $0 \leq j \leq n/2-1$. For every j, $0 \leq j \leq n/2-1$ and $l = 1,\ldots,\Delta$, there is an edge with the label (weight) l between the vertices $(1,j)$ and $(2,(j+2^{l-1}-1) \bmod n/2)$. The edges with the given label l are said to be in dimension l.*

Note that $W_{1,n}$ consists of $n/2$ disconnected edges. For $\Delta \geq 2$, $W_{\Delta,n}$ is connected.

In this paper we give the results of some new observations, concerning a slight modification of Knödel graphs, obtained by a cyclic permutation of the dimensions of the edges. For the graph under consideration, the modification looks as follows: dimension of the edge connecting the vertices $(1,j)$ and $(2,(j+2^{l-1}-1) \bmod n/2)$, is replaced by $(l+p) \bmod \lceil \log_2 n \rceil$, where $p = 1,\ldots,\Delta$. For the case of $n = 2^k - 2$, we already showed the ability of Knödel graphs to preserve gossiping properties for the operation described in ([13]). In this paper we consider general case of $n \neq 2^k$, and show the ability of Knödel graphs to preserve gossiping after the cyclic permutation of the dimensions of its edges. We have shown in [13] that in case of $n = 2^k$, generally, the obtained graphs have no gossipping properties at all.

2. DEFINITIONS AND NOTATIONS

In this section we consider the modified Knödel graphs with the number of vertices not equal to 2^k, where k is any integer with $k \geq 3$. In what follows, we give the definition of the modified Knödel graph.

Definition 2. *The modified Knödel graph on $n \geq 2$ vertices (n even) and of maximum degree $\Delta \geq 1$ is denoted by $M_{\Delta,n}(p)$. The vertices of $M_{\Delta,n}(p)$ are the pairs (i,j) with $i = 1,2$ and $0 \leq j \leq n/2-1$, respectively. For every j, $0 \leq j \leq n/2-1$ and $l = 1,\ldots,\Delta$, there is an edge with the label $(l+p) \bmod \lceil \log_2 n \rceil$ between the vertices $(1,j)$ and $(2,(j+2^{l-1}-1) \bmod n/2)$, where* *p is a fixed integer for the modified graph with possible values $p = 1, \ldots, \Delta$. The edges with the label $l + p$ are said to be of dimension $l + p$.*

From now on, we will be using the terms "modified" and "edge-permuted" interchangeably. According to the above definition, there exist $\Delta - 1$ modifications for each $W_{\Delta,n}$. In this paper we consider only the case, when $\Delta = \lceil \log_2 n \rceil$, and this value will be assumed hereafter for all references of Δ.

Definition 3. *A path $P = (u, e_1, v_1, e_2, v_2, \ldots, e_k, v)$ between vertices u and v in a weighted graph is said to be ascending if for each e_i and e_j $t_G(e_i) < t_G(e_j)$ for $1 \le i < j \le k$.*

Definition 4. *Consider two graphs, $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with the same set of vertices V and labeled edge sets E_1 and E_2, respectively. The edge sum of these graphs is a new graph $G_1 + G_2 = G = (V, E)$ with $E = E_1 \cup E_2$, where $e \in E$ are labeled by the following rules:*

$$t_G(e) = \begin{cases} t_{G_1}(e), & \text{if } e \in E_1, \\ t_{G_2}(e) + \max\limits_{e' \in E_1} t_{G_1}(e'), & \text{if } e \in E_2. \end{cases} \quad (1)$$

where $t_G(e)$ is the label of the edge e in the resulting graph G.

It is known that if $G = W_{\Delta,n} + W_{1,n}$, then G is a complete gossip graph.

From now on, when mentioning the vertex $u = (i_1, j_1)$, we assume that $i_1 = 1$ and $j_1 = 0$. Since the Knödel graphs are symmetric, the same assumptions or statements are true for every other vertex $u' = (1, j')$.

Definition 5. *The distance from the vertex $u = (i_1, j_1)$ to the vertex $v = (i_2, j_2)$ of the Knödel graph where $(j_1 + 2^{\lfloor \log_2 n \rfloor}) \bmod n/2 \ge j_2$ can be calculated by the following formula:*

$$D_{u,v} = |i_1 - i_2| + |j_1 - j_2| \quad (2)$$

It is obvious that for each vertex u of the graph there exist two vertices $v_1 = (i_1, j_1)$ and $v_2 = (i_2, j_2)$, such that $D_{u,v_1} = D_{u,v_2}$ and $i_1 \ne i_2$. Let us consider the vertices v_1 and v_2 to be "clones" of each other. As we have already introduced the modified Knodel graph and the distance between its vertices, we can now define a "base" for this graph.

Definition 6. *The "base" of the modified Knödel graph, relative to the vertex $u = (i_1, j_1)$, are all the vertices v, such that $0 \le D_{v,u} < \lfloor \log_2 n \rfloor$.*

Let the vertex u be called the "root" of the base.

Let us now consider the path between the vertex u and some vertex v that belongs to the "base" of the modified Knödel graph relative to the vertex u, in the original Knödel graph. For that purpose, first we represent the distance defined in (2) in the base-two system. Next, we transform the base-two representation (2) into a sum that will result to the original distance (in base-ten system), also to a path representation between two vertices.

Definition 7. *Let $Path_{u,v}$ be equal to the sum resulting in $D_{u,v}$, the terms of which are represented by different powers of two:*

$$Path_{u,v} = \sum_{l=0}^{k} s_l * (2^{l+1} - 2^l), \quad (3)$$

where $k = \lfloor D_{u,v} \rfloor$ and $s_l = 0$ or 1, depending on the base-two representation of the $D_{u,v}$.

By simplifying (3), we get a new expression for the $Path$, as follows: $Path_{u,v} = 2^{k+1} - 2^k + 2^{k-1} \ldots \pm 2^0$. Note, that it can appear that the $Path_{u,v}$ presents the path between the vertex u and the clone of the vertex v. Will come back to this case and discuss it separately.

Let us now define the transformation on the $Path_{u,v}$ that will result in a completely new $Path_{u,v}$ which will show the ascending path between any pairs of vertices u' and v, where u' is the vertex different from the original destination vertex u and is located inside the "base" of the edge-permuted Knödel graph.

Definition 8. *The operation $T(Path_{u,v}) = T(2^{k+1} - 2^k + 2^{k-1} \ldots \pm 2^m) = Path_{M(u,2^m),v} = (Path_{u,v} - 2^m)$, where $Path_{u,v} - 2^m$ is a new path obtained as a result of the transformation T on the $Path_{u,v}$, $M(u, 2^m)$ is a new vertex such that $D_{u,M(u,2^m)} = 2^m$, where m is the exponent of the smallest member of the $Path_{u,v}$.*

Figure 1 below illustrates the result of the application of the operation T on the highlighted path of the edge-permuted Knödel graph that results in a new path between the vertices 1 and 6.

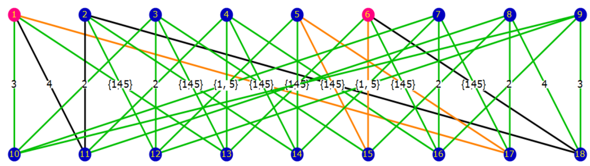

Figure 1. Path transformation

We will apply the transformation T on the 1-folded ascending path $6 \to 1$ (original definition is as follows: $Path_{1,6} = (2^3 - 2^2) + (2^1 - 2^0)$) in order to find a new path that will have no folds at all. Since $T(Path_{1,6}) = Path_{2,6} = 2^3 - 2^2 + 2^1 - 2^0 - 2^0 = 2^3 - 2^2$, the obtained path has no folds in the edge-permuted graph, and presents the path between the vertices 6 and 2 (colored black in Fig. 1). Obviously, there exists an ascending path in the "base" of the edge permuted Knödel graph that connects the vertices 2 and 1 and has higher weights than that of the edges of the path $6 \to 2$. Thus, we can insist that the concatenation of these two paths is an ascending path between the vertices 6 and 1.

3. MODIFIED KNÖDEL GRAPHS

In this section we consider the edge-permuted Knödel graphs (or the Modified Knödel graphs) and show that the gossiping properties of these graphs remain the same after the edge permutation. The main goal here is to show that if there is an ascending path between the given two vertices in the original Knödel graph, then there should be such a path in the edge-permuted version of that graph. This is obtained by showing that there exists at most 1-folded ascending path from every vertex to a given vertex $(1, 0)$ in the modified Knödel graph.

Lemma 1. There exists an ascending path between any two vertices u and v $(v \rightarrow u)$ of the Knödel graph, if $D_{u,v} \leq 2^{\lfloor log_2 n \rfloor}$.

Proof. It is known that Knödel graphs have descending tree rooted at each node $u = (i_1, j_1)$ and containing each node $v = (i_2, j_2)$, where $(j_1 + 2^{\lfloor log_2 n \rfloor}) \bmod n/2 \geq j_2$. Since $D_{u,v} \leq 2^{\lfloor log_2 n \rfloor}$, it follows that the vertex v belongs to a descending tree rooted at u, and hence, there exists an ascending path from v to u. □

Lemma 2. There exists at most 1-folded ascending path between any two $u = (i_1, j_1)$ and $v = (i_2, j_2)$ vertices $(v \rightarrow u)$ of the edge-permuted Knödel graph in case $(j_1 + 2^{\lfloor log_2 n \rfloor}) \bmod n/2 > j_2$.

Proof. There exists a path in the edge-permuted Knödel graph between any pairs of vertices $u = (i_1, j_1)$ and $v = (i_2, j_2)$ $(v \rightarrow u)$ in case $j_1 + 2^{\lfloor log_2 n \rfloor}) \bmod n/2 > j_2$ which is either 0 or 1 folded, depending on the position of the vertex v. Note, the path is the same as in the original Knödel graph, only the weights of edges are different. Since we have increasing order in the new "base" of the edge-permuted Knödel graph, we also have 0 folded ascending path between any vertex of the "base" and the vertex u. If the vertex v is located outside the new "base" of the edge-permuted Knödel graph, but at the same time, $j_2 < (j_1 + 2^{\lfloor log_2 n \rfloor}) \bmod n/2$, then there exists at most 1-folded ascending path connecting the vertices v and u. The folding point of this path is located outside the "base" of the edge-permuted Knödel graph, but inside the range $(j_1, (j_1 + 2^{\lfloor log_2 n \rfloor}) \bmod n/2)$. Indeed, in case we have a 1-folded ascending path, we can classify the edges of this path into two groups:
1) the weights of the edges belong to the "base" of the graph 2) otherwise.

Obviously, the point where the two subpaths of the initial path are joined is also a folding point for the whole path. In the example below, the vertex 15 is the folding point for the path $6 \rightarrow 1$. Finally, by applying the transformation T to the path, we will be able to find a corresponding ascending path (without any folds) between these two vertices. □

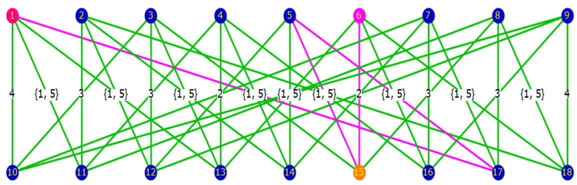

Figure 2. Folding point of the path $6 \rightarrow 1$

Lemma 3. There exists an ascending path from any vertex in the base to the root vertex.

Proof. The above lemma follows from the fact that we have an ascending edge order in the scope of the base (like in the original Knödel graph), so there exists an ascending tree rooted at the root vertex of the base and containing all the vertices of the base. □

Let us now return to the $Path$ defined in the previous section and give the modification that will make it possible to transform the $Path$ in a way to obtain the path to the "clone" vertex of the source vertex (since in this scope we consider descending paths). Let us denote the result of this modification by $Path'$. We can assume that $Path$ presents a descending path from the vertex u to the vertex v_1, so from the notation above, $Path_{u,v_1} = Path'_{u,v_2}$, where $D_{v_1} = D_{v_2}$. Firstly, assume that $Path_{u,v_1}$ has the following form: $2^{k+1} - 2^k + 2^{k-1} ... \pm 2^m$.

Lemma 4. $Path'_{u,v_2} = Path_{u,v_1} \pm (2^m - 2^m)$ shows the descending path from the vertex u to the vertex v_2.

Proof. The above lemma is obvious, since after applying the modification described, the last component of the $Path'_{u,v_2}$ gets an inverse sign, comparing with that of $Path_{u,v_1}$, hence the last vertex is in the opposite side of the obtained bipartite graph, also the absolute values $Path_{u,v_1}$ and $Path'_{u,v_2}$ are equal, and therefore, the distances of v_1 and v_2 from the vertex u are also equal. □

Theorem 1. The graph $G = M_{\Delta,n}(p) + M_{1,n}(1)$ is a gossip graph, for any $n \neq 2^k$ and $p = 1, \ldots, \Delta$.

Proof. From the lemmas $(1) - (4)$ it follows that there exists an ascending path between any two vertices $u = (i_1, j_1)$ and $v = (i_2, j_2)$ on the edge-permuted Knödel graph in case $(j_1 + 2^{\lfloor log_2 n \rfloor}) \bmod n/2 > j_2$. The only case, left out from consideration, is when the expression $(j_1 + 2^{\lfloor log_2 n \rfloor}) \bmod n/2 > j_2$ does not hold true. For those cases we should consider the graph $M_{1,n}(1)$ and the vertex $u' = (i'_1, j'_1)$ which is connected to the vertex u by the edge of the graph $M_{1,n}(1)$. Let us denote the edge of $M_{1,n}(1)$ connecting the vertices u' and u by t'. Since the edges of $M_{1,n}(1)$ have the highest weights among all the edges of the graph, it follows that t' is the last edge of all the paths going to u through the vertex u'. Since this graph is symmetric, it can be shown (as we showed above) that there exists an ascending path from every vertex $(j_1 + 2^{\lfloor log_2 n \rfloor}) \bmod n/2 < j_2$ to the vertex u', and these paths will then reach the vertex u through the edge t'. □

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