Functional System of the Fuzzy Constructive Logic

Igor D. Zaslavskiy
Institute for Informatics and Automation Problems of NAS of RA
Yerevan, Armenia
e-mail: zaslav@ipia.sci.am

ABSTRACT

The principles of constructive mathematics are applied to the fuzzy constructive logic. The preceding results of the author in fuzzy constructive logic are generalized for predicate formulas including (in general) functional symbols and symbols of constants. The constructive (intuitionistic) predicate calculus on the base of such formulas is considered; it is denoted by $H^{(f\ con)}$. The notion of identically f-true predicate formula is introduced. It is proved that any predicate formula deducible in $H^{(f\ con)}$ is identically f-true.

Keywords

Predicate calculus, functional symbol, functional assignment, recursive function, fuzzy set, fuzzy ideal.

The considerations below are based on the principles of constructive mathematics ([1]-[3]); A. A. Markov's principle [4] will not be used. We will consider the concepts of fuzzy constructive logic ([5]-[8]).

By N we denote the set of non-negative integers, $N = \{0, 1, 2, \dots\}$, by N^n we denote the set of n-tuples $(x_1, x_2, \dots, x_n)$, where $x_i \in N$ for $1 \le i \le n$; by R we denote the set of rational numbers having the form $\frac{m}{2^n}$, where $m \in N$, $n \in N$, $0 \le \frac{m}{2^n} \le 1$. The notions of general recursive function (shortly, GRF), partially recursive function (shortly, PRF), recursively enumerable set (shortly, RES) are defined as in ([9]-[11]).

The considerations below will be based on the notion of fuzzy recursively enumerable set (FRES) given in [6] (see [6], p. 38). Let us recall that n-dimensional FRES, where $n \ge 1$, is defined as a recursively enumerable set of $(n+1)$-tuples having the form $(x_1,\ x_2, \dots, x_n, \varepsilon)$, where $x_i \in N$ for $1 \le i \le n$ and $\varepsilon \in R$. (Let us note that the notion of FRES is actually equivalent to the notion of generalized fuzzy recursively enumerable set (GFRES) concerning the algorithmic scale of truth values (A-scale) Ω_2 considered in [7]; this equivalence is established in [8]. Let us note also that the relations between FRESes and the operations on FRESes considered below are similar to those given in [7]; they are in general different from the relations and operations considered in [6]).

We say that an n-dimensional FRES α covers an n-dimensional FRES β, and write $\beta \subseteq \alpha$ or $\alpha \supseteq \beta$, if for any $(n+1)$-tuple $(x_1,\ x_2, \dots, x_n, \varepsilon) \in \beta$, where $\varepsilon \ne 0$, there exists an $(n+1)$-tuple $(x_1, x_2, \dots, x_n, \delta) \in \alpha$ such that $\delta \ge \varepsilon$. We say that n-dimensional FRESes α and β are equivalent and write $\alpha = \beta$ if α covers β and β covers α. The negations of statements $\alpha \supseteq \beta$ and $\alpha = \beta$ will be denoted by $\alpha \not\supseteq \beta$ and $\alpha \ne \beta$.

Note. The notions of covering and equivalence of FRESes given above coincide with the corresponding notions in [7]; they are different from the notions given in [6].

We say that an n-dimensional FRES α is monotone if for any $(n+1)$-tuples $(x_1,\ x_2, \dots, x_n, \varepsilon)$ and $(x_1,\ x_2, \dots, x_n, \delta)$ the folowing condition is satisfied: if $(x_1,\ x_2, \dots, x_n, \delta) \in \alpha$ and $\varepsilon \le \delta$, then $(x_1,\ x_2, \dots, x_n, \varepsilon) \in \alpha$.

An n-dimensional FRES α is said to be regular if any $(n+1)$-tuple having the form $(x_1,\ x_2, \dots, x_n, 0)$ belongs to α (cf. [8]).

The union $\alpha \cup \beta$ (correspondingly, the intersection $\alpha \cap \beta$) of n-dimensional FRESes α and β is defined as an n-dimensional FRES γ such that $(x_1,\ x_2, \dots, x_n, \eta) \in \gamma$ if and only if there exist $\varepsilon \in R$ and $\delta \in R$ such that $(x_1,\ x_2, \dots, x_n, \varepsilon) \in \alpha$, $(x_1,\ x_2, \dots, x_n, \delta) \in \beta$, $\eta \le \max(\varepsilon, \delta)$ (correspondingly, $\eta \le \min(\varepsilon, \delta)$) (cf. [6], p. 41; [7], p. 270).

By V^n we denote the n-dimensional FRES α such that $(x_1,\ x_2, \dots, x_n, \varepsilon) \in \alpha$ for any n-tuple $(x_1,\ x_2, \dots, x_n) \in N^n$ and any $\varepsilon \in R$. By Λ^n we denote the n-dimensional FRES β such that $(x_1,\ x_2, \dots, x_n, \varepsilon) \in \beta$ if and only if $\varepsilon = 0$.

The operation of Cartesian product of the n-dimensional FRES α and m-dimensional FRES β we will consider only in the case when $\beta = V^m$ (only such case is used in what follows) (cf. [8]). Namely, $\alpha \times V^m$ is defined by the set of $(n+m)$-tuples having the form $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m, \varepsilon)$, where $(x_1,\ x_2, \dots, x_n, \varepsilon) \in \alpha$, $y_j \in N$ for $1 \le j \le m$.

The following notions are defined similarly to the corresponding definitions given in [7] (see [7], pp. 270-271):

1) the notion of fictitious variable for a FRES α;
2) the notion of projection $\downarrow_i^n(\alpha)$ of an n-dimensional FRES α concerning i-th coordinate;
3) the notion of generalization of a FRES α concerning i-th coordinate;
4) the notion of transposition $T_{ij}^n(\alpha)$ of i-th and j-th coordinates in an n-dimensional FRES α;
5) the notion of substitution $Sub_{ij}^n(\alpha)$ of i-th coordinate for j-th coordinate in an n-dimensional FRES α (see also [6], p. 42).

n-dimensional FRES-ideal is defined as a non-empty constructive set Δ of n-dimensional monotone regular FRESes such that the following conditions hold:

(1) if $\alpha \in \Delta$ and $\beta \subseteq \alpha$, then $\beta \in \Delta$;
(2) if$\alpha \in \Delta$ and $\beta \in \Delta$, then $\alpha \cup \beta \in \Delta$.

Note. The notion of FRES-ideal corresponds to the notion of ideal given in [7]; it is different from the notion of ideal considered in [6].

An n-dimensional FRES-ideal Δ is said to be complete if any n-dimensional FRES of the n-tuples $(x_1,\ x_2, \dots, x_n, \varepsilon)$, where $0 \leq \varepsilon < 1$, belongs to Δ. It is said to be null-ideal if $\alpha \in \Delta$ only when $\alpha \in \Lambda^n$.

An n-dimensional FRES-ideal generated by some non-empty set of n-dimensional monotone regular FRESes is defined similarly to the definition given in [6] (see [6], pp. 46-47).

We consider predicate formulas based on the logical operations $\&, \vee, \supset, \neg, \forall, \exists$ and containing in general functional symbols and symbols of constants ([9]-[11]). We suppose that the language of predicate formulas contains an infinite set of symbols of constants as well as infinite sets of predicate and functional symbols having all positive dimensions. It is supposed that the symbol T of the truth and the symbol F of the falsity are included in the language.

The notion of index majorant of a given predicate formula is defined as in [7] (see [7], p. 272).

The constructive (intuitionistic) predicate calculus (on the base of formulas containing in general functional symbols and symbols of constants) is defined as in [12]. This calculus will be denoted below by $H^{(f\ con)}$.

Let A be a predicate formula satisfying the following conditions.
1) All predicate symbols included in A belong to the list $p_1, p_2, \dots, p_n$ of symbols having the dimensions $k_1, k_2, \dots, k_n$.
2) All functional symbols included in A belong to the list $f_1, f_2, \dots, f_m$ of symbols having the dimensions $l_1, l_2, \dots, l_m$.
3) All symbols of constants included in A belong to the list $a_1, a_2, \dots, a_r$.

Functional assignment (shortly, f-assignment) φ for a given predicate formula A satisfying the conditions mentioned above is defined as a correspondence giving for any predicate symbol p_i, any functional symbol f_j, any symbol a_t of constant their φ-values such that the following conditions hold.
1) The φ-value of any predicate symbol p_i having the dimension k_i is a k_i-dimensional FRES-ideal.
2) The φ-value of any functional symbol f_j having the dimension l_j is a GRF depending on l_j variables.
3) The φ-value of any symbol of constant a_t is an element of the set N (i.e., a non-negative integer).

The interpretation of a formula A satisfying the conditions mentioned above concerning an f-assignment φ and an index majorant k is defined as a FRES-ideal $\Pi_{\varphi,k}(A)$ by induction following to the construction of the formula A as follows.

If B is an elementary subformula of A having the form $p_i(t_1, t_2, \dots, t_m)$, where $1 \leq i \leq n$, $m = k_i$ and $t_1, t_2, \dots, t_m$ are terms containing general functional symbols, symbols of constants and object variables, then $\Pi_{\varphi,k}(B)$ is obtained using the following constructions. Let h be the maximum of the indices i of object variables x_i contained in the terms $t_1, t_2, \dots, t_n$. The φ-values of the terms $t_1, t_2, \dots, t_n$ may be represented (adding, if it is necessary, additional fictitious object variables) as GRF, correspondingly $\psi_1, \psi_2, \dots, \psi_n$ depending on variables $x_1, x_2, \dots, x_h$. If α is any n-dimensional FRES, then by $\alpha(\psi_1, \psi_2, \dots, \psi_n)$ we denote the recursively enumerable set of $(h+1)$-tuples $(x_1, x_2, \dots, x_h, \varepsilon)$ such that $(x_1, x_2, \dots, x_h, \varepsilon) \in \alpha(\psi_1, \psi_2, \dots, \psi_n)$ if and only if $(\psi_1(x_1, x_2, \dots, x_h), \psi_2(x_1, x_2, \dots, x_h), \dots, \psi_n(x_1, x_2, \dots, x_h), \varepsilon) \in \alpha$.

Now let Δ be the φ-value of p_i. By Δ^* we denote the h-dimensional FRES-ideal generated by the set of FRESes having the form $\alpha(\psi_1, \psi_2, \dots, \psi_n)$, where $\alpha \in \Delta$. The FRES-ideal $\Pi_{\varphi,k}(A)$ is defined as the k-dimensional FRES-ideal generated by the set of FRESes having the form $\beta \times V^{k-h}$ (obviously, if k is an index majorant of the formula A, then $k \geq h$).

If B and C are subformulas of A, then the FRES-ideals $\Pi_{\varphi,k}(B\&C)$, $\Pi_{\varphi,k}(B \vee C)$, $\Pi_{\varphi,k}(B \supset C)$, $\Pi_{\varphi,k}(\forall x_i(B))$, $\Pi_{\varphi,k}(\exists x_i(B))$, $\Pi_{\varphi,k}(\mathrm{T})$, $\Pi_{\varphi,k}(\mathrm{F})$ are defined similarly to the corresponding definitions given in [6] and [7] (see [6] p. 51; [7] p. 273).

In particular, by induction also the FRES-ideal $\Pi_{\varphi,k}(A)$is defined.

We say that a formula A having the form mentioned above is identically f-true if for any f-assignment φ and any sufficiently great index majorant $k\Pi_{\varphi,k}(A)$ is a complete FRES-ideal.

Theorem. Any predicate formula deducible in the calculus $H^{(f\ con)}$ is identically f-true.

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