A Linear Algebra Approach to Some Problems of Graph Theory

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ABSTRACT

In this paper, we consider some known problems of graph theory from the linear algebra point of view. Studying features of vector spaces over $GF(2)$ allows us to reprove the theorem on graph circuits and cut-sets and develop a new algorithm to recognize a line graph and construct its original graph.

Keywords

Vector spaces over $GF(2)$, graph circuits and cut-sets, line graph recognition.

1. INTRODUCTION

The study of graphs using a linear algebra approach gives many interesting results. For example, the theorems relating spectral properties of the adjacency matrix to other graph properties (see, e.g., [3]), the famous Cheeger's inequality approximating the sparsest cut-set (one of the most useful facts in algorithmic applications) [9], the theorems relating graph diameter and eigenvalues [1, 12] were proved. It is worth mentioning that the well-known graph isomorphism problem can be treated as a linear algebra problem [6].

In this paper, we consider n-dimensional vector spaces over the field $GF(2)$ consisting of two elements, 0 and 1, with operations (addition and multiplication) using the usual operations on integers, followed by reduction modulo 2. With the help of special properties of these vector spaces, we prove the theorem on graph circuits and cut-sets [15] and present a new algorithm to recognize a line graph and construct its original graph [5, 10, 11, 13]. The problem of line graph recognition is important because some practical problems of graph theory have rather simple solutions for line graphs [5].

In our opinion, the proposed approach can be applied to solve some other problems. For example, it can be used in analysis of vertex and edge covers of graphs (see, e.g, [8, 15]).

Further, we consider only *undirected* and *simple* graphs.

2. THE THEOREM ON GRAPH CIRCUITS AND CUT-SETS

The following theorem is well-known in graph theory [4, 15, 16].

Theorem 1. Every graph G can be expressed as a ring sum of two subgraphs, one of which is in the circuit subspace and the other is in the cut-set subspace of G.

Further, we obtain this result examining a vector space over $GF(2)$, and its orthogonal complement.

Consider a vector space $\mathbb{G}^n$ composed of n-tuples of elements of $GF(2)$ (or $(0,1)$-vectors), such as

$$X = (x_1, x_2, \ldots, x_n)^T,$$

$$\text{where } x_j \in \{0,1\}, \text{ for } j = 1, 2, \ldots, n$$

with component-wise addition and multiplication by an element from $GF(2)$.

Consider linearly independent vectors $X_1, X_2, \ldots, X_m \in \mathbb{G}^n$. Denote by L the linear span of these vectors. Let vectors $Y_1, Y_2, \ldots, Y_{n-m}$ be a basis for the orthogonal complement $L^\perp$. Denote by J the vector $(1, 1, \ldots, 1)^T$. We have the following theorem [7].

Theorem 2. The vector J is a linear combination of vectors $X_1, \ldots, X_m$, Y_1, …, Y_{n-m}.

Proof. For every vector $Z \in L \cap L^\perp$, we have $(Z, Z) = 0$ (here (X,Y) stands for the inner product of two vectors X and Y). Hence, all such vectors Z are even, i.e., they have even nonzero components. Consequently, $(Z, J) = 0$ and $J \in (L \cap L^\perp)^\perp$.

Suppose $\dim(L \cap L^\perp) = k$. Then $\dim(L \cap L^\perp)^\perp = n - k$. Besides this,

$$\dim(L + L^\perp) = \dim L + \dim L^\perp - \dim(L \cap L^\perp) = n - k.$$

If any vector $V \in L + L^\perp$, then $V = X + Y$, where $X \in L, Y \in L^\perp$. The vector X is orthogonal to every vector of $L^\perp$. Hence, X is orthogonal to every vector of $L \cap L^\perp$. Therefore, $X \in (L \cap L^\perp)^\perp$. Similarly, $Y \in (L \cap L^\perp)^\perp$. This proves that $V \in (L \cap L^\perp)^\perp$. Hence, $L + L^\perp \subseteq (L \cap L^\perp)^\perp$. Since

$$\dim(L \cap L^\perp)^\perp = \dim(L + L^\perp),$$

it follows that $L + L^\perp = (L \cap L^\perp)^\perp$. □

In fact, more is proven here than just the desired claim. We get that every vector in $(L \cap L^\perp)^\perp$ is a linear combination of vectors $X_1, \ldots, X_m$, Y_1, …, Y_{n-m}.

The proof of Theorem 1 in [4, 16] is much more complicated and longer than the proof based on the theory of vector spaces over $GF(2)$ presented here.

3. PROBLEM OF LINE GRAPH RECOGNITION

Now we consider the problem of recognizing a line graph and constructing its original graph.

3.1 Theoretical preliminaries

We assume that a given graph is not a complete graph on three vertices (in this exceptional case there exist two non-isomorphic original graphs). In all other cases, if the line graphs of two connected graphs are isomorphic, then the original graphs are isomorphic [17].

There are three algorithms to determine if a given graph is a line graph [5, 10, 13]. Recently, a new effective algorithm has been proposed [11]. Here we provide a linear algebraic approach to solve this problem. The algorithm is based on the following theorem [8]:

Theorem 3. *A graph G is a line graph if and only if the edges of G can be partitioned into cliques in such a way that every vertex belongs to exactly two of the cliques and no two vertices of G are both in the same two cliques.*

Remark 1. *It is worth mentioning that a clique may consist of only one vertex. Obviously, such a clique has no edges.*

Given such a partition into cliques, the original graph H for which G is the line graph can be constructed by assigning one vertex in H to each clique, and an edge in H to each vertex in G with its endpoints being the two cliques containing the vertex in G.

The problem of line graph recognition is related to that of matrix factorization. In fact, let D be the adjacency matrix of a given graph G on m vertices. We have the following theorem.

Theorem 4. *A graph G is the line graph of a certain graph H if and only if there exists a $(0,1)$-matrix B containing exactly two units in every its column such that $D = B^T B$. In addition, B is the incidence matrix of the graph H.*

Proof. Necessity. According to Theorem 3, partition the edges of G into n cliques $C_1, C_2, \dots, C_n$. To each clique C_k assign the column $\mathfrak{B}_k \in \mathbb{G}^m$ such that its element in i-th position equals 1 if and only if the i-th vertex belongs to the clique C_k. Consider the $m \times n$ matrix $B^T = (\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n)$.

Since every vertex of G lies in exactly two of the cliques, then every column B_i $(i \in \{1, 2, \dots, m\})$ of the matrix B contains exactly two units. Since any two different cliques have not more than one common vertex, all the columns of the matrix B are different. Thus, the matrix B can be treated as an incidence matrix of some graph H with n vertices that is the original graph for G. Really, the k-th row of matrix B corresponds to the clique C_k. The j-th and k-th vertices of G are adjacent if and only if the inner product of the corresponding rows of matrix B is equal to 1.

Sufficiency. Let B be the incidence matrix of a given graph H and D be the adjacency matrix of the corresponding line graph $G = L(H)$. The relation $D = B^T B$ is well-known [14]. It can be proved by direct calculations. □

Corollary 1. *Let a graph G with adjacency matrix D be a line graph of a graph H with incidence matrix B, $D = B^T B$. Then there exists a row partition of matrix B^T, such that the corresponding subgraphs of graph $G = L(H)$ with matrix D are cliques satisfying the conditions of Theorem 3.*

Proof. Denote by N_k $(k = 1, 2, \dots, n)$ the set of rows of matrix B^T that have unit in the k-th position. Obviously, this generates the desired row partition of the matrix B^T. □

3.2 Algorithm description

Let us describe an algorithm to recognize a line graph that is based on the approach provided in [10]. According to Theorem 4, for a given graph G with the adjacency matrix $D = [d_{ij}]_{m \times m}$, we want to find a $(0,1)$-matrix B, such that $D = B^T B$. In addition, in every column of B there must be exactly two units.

Consider the first vertex of the graph G. Assign this vertex to two cliques with numbers 1 and 2. Hence, the first row of the matrix B^T is of the form

$$(1, 1, 0, \dots, 0).$$

Note that we do not know the number of columns of B^T, but it is obvious that this number does not exceed $m + 1$.

If vertices 1 and 2 are adjacent (since $d_{12} = 1$), then we assign vertex 2 to cliques 1 and 3. In the opposite case (when $d_{12} = 0$), we assign vertex 2 to cliques 3 and 4.

Every vertex of G must be assigned to two cliques. This assignment procedure is unique up to the numeration of the cliques with only one exception. The exceptional case happens when there exist three pair-wise adjacent vertices. In this case, we need in two additional definitions. Then we prove Theorem 5 that allows us to assign the vertices to cliques uniquely.

Definition 1. *Given two adjacent vertices of G, there are two corresponding edges in H that share a vertex. Let us assume that the other endpoints of these edges are incident to an edge. Then the vertex of G that corresponds to this edge is called the* cross node *[10].*

Without loss of generality, we assume that vertices 1, 2, and 3 are pair-wise adjacent. Then we assign vertex 1 to cliques 1 and 2, and vertex 2 to cliques 1 and 3. If vertex 3 is adjacent to the considered vertices, it may be assigned to cliques 1 and 4, or to cliques 2 and 3. The choice of one of these two alternatives is defined by elements of matrix D.

Definition 2. *We say that* the evenness condition is fulfilled, *if in the first three rows of matrix D the number of units in every column is even.*

In other words, the evenness condition is fulfilled if and only if the triangle with vertices 1, 2, and 3 is even, i.e., every vertex of G is adjacent to even vertices of these three ones.

Theorem 5. *The evenness condition is fulfilled if and only if there exists a cross node.*

Proof. Denote by I, II, III, IV, and so on, the edges of H corresponding to vertices of G with numbers 1, 2, 3, 4, and so on, respectively.

Necessity. Suppose that the evenness condition is satisfied. We show that in this case there exists a cross node.

Recall that we do not consider the exceptional case when there are no other vertices adjacent to vertices 1, 2 and 3, although the evenness condition is fulfilled. Consider one of the edges of H that shares common vertices with two edges of edges I, II, III. Without loss of generality, suppose edge IV shares vertices with edges I and II. Assume that vertex 3 is not a cross node, so the edges I, II and III share an endpoint. Then we immediately obtain that vertex 4 is a cross node.

Sufficiency. Obviously, if vertex 3 is a cross node, then the evenness condition is fulfilled. Really, in this case every edge of H shares a vertex with even number of three edges I, II, III. □

According to Theorem 5, in all cases, when the evenness condition is fulfilled, we assign vertex 3 to cliques 2 and 3, otherwise – to cliques 1 and 4. It is clear that the matrix B^T is unique. In this manner, the rest of the vertices are uniquely assigned to the corresponding cliques.

If there exists a vertex that cannot be assigned to two different cliques when all adjacency conditions are fulfilled, then the graph is not a line graph.

If matrix B^T is computed, then the graph is a line graph, and we can find the matrix A of its original graph H by the formula $A = BB^T + S$, where S is a diagonal matrix that has units at the same places, as BB^T. The number of columns of matrix B^T is uniquely defined.

3.3 The algorithm

In what follows, we denote by C_i the i-th clique, $|C_i|$ — the number of elements in C_i, P_i — one-dimensional array containing the i-th row of adjacency matrix D, n_{cl} — the number of cliques containing the current vertex.

```
Input: an adjacency matrix D
Output: the incidence matrix of original graph (if it exists)
begin  Vertex 1 assign to cliques C_1 and C_2
for i = 2 to m
    n_cl = 0
    for k = 1 to m + 1
      if the i-th vertex is adjacent to all the vertices in C_k or |C_k| = 0 then
        if |C_k| ≠ 2 or |C_k| = 2 and the evenness condition is not fulfilled
        then
          the i-th vertex assign to C_k
          n_cl = n_cl + 1
          if n_cl = 2 then goto M1 end
          set to 0 all the components of P_i corresponding to the vertices that
              are contained in C_k
        end
      end
    M1: end for k
    if n_cl < 2 then
      The graph is not a line graph
      goto M2
    end
end for i
The graph is a line graph
M2: end
```

3.4 Examples

The first example shows how the algorithm works in the case of a line graph, and in the second example we consider a graph that is not a line one.

Example 1. *Consider a graph G with the adjacency matrix*

$$D = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The first row of B^T is as follows:

$$(1, 1, 0, \ldots, 0).$$

The first two vertices are adjacent, so the second row of B^T is as follows:

$$(1, 0, 1, 0, \ldots, 0).$$

The third vertex is adjacent to the first two ones, and the evenness condition is not fulfilled. Hence, we get the third row of B^T

$$(1, 0, 0, 1, 0, \ldots, 0).$$

The fourth vertex is adjacent to the first and third ones, so the fourth row is

$$(0, 1, 0, 1, 0, \ldots, 0).$$

The other rows of B^T are defined in a similar way. We arrive at matrix

$$B^T = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Now we can find matrix A of graph H that is the original graph for G:

$$A = BB^T + S = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Example 2. *Consider a graph G with the adjacency matrix*

$$D = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

This graph is one of Beineke's forbidden-subgraph characterizations of line graphs [2].

The first row of B^T is as follows:

$$(1, 1, 0, \ldots, 0).$$

The first two vertices are adjacent, so the second row of B^T is as follows:

$$(1, 0, 1, 0, \ldots, 0).$$

The third vertex is non-adjacent to the first two. Hence, the third row is as follows:

$$(0, 0, 1, 1, 0, \ldots, 0).$$

The fourth vertex is adjacent to the first two vertices and non-adjacent to the third one. The evenness condition for vertices 1, 2 and 4 is not fulfilled. In rows 1, 2 and 4 there is only one unit in the third column of D . Hence, the fourth row is

$$(1, 0, 0, 0, 1, 0, \ldots, 0).$$

The fifth vertex is adjacent to vertices 2, 3 and 4, and non-adjacent to the first vertex. Therefore, the fifth row is as follows:

$$(0, 0, 1, 0, 1, 0 \ldots, 0).$$

And the sixth vertex is adjacent to vertices 3 and 5, and non-adjacent to the other ones. This means that all elements in columns 1, 2, 3 and 5 in the sixth row are zeros. This condition cannot be fulfilled because the sixth and the fifth rows must be the same.

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REFERENCES

[1] N. Alon, V. D. Milman, "λ_1, Isoperimetric Inequalities for Graphs, and Superconcentrators", *J. of Combin. Theory, Ser. B* **38**, pp. 73–88, 1985.

[2] L. W. Beineke, "Characterizations of derived graphs", *Journal of Combinatorial Theory* **9** no. 2, pp. 129–135, 1970.

[3] N. Biggs, "Algebraic Graph Theory (2nd ed.)", Cambridge: Cambridge University Press, 1993.

[4] W.-K. Chen, "On vector spaces associated with a graph", *SIAM J. Appl. Math.* **20** no. 3, pp. 526–529, 1971.

[5] D. G. Degiorgi, K. Simon, "A dynamic algorithm for line graph recognition", *Lecture Notes in Comput. Sci.* **1017**, pp. 37–48, 1995.

[6] Sh. Friedland, "Quadratic Forms and the Graph lsomorphism Problem", *Linear Algebra Appl.* **150**, pp. 423–442, 1991.

[7] P. R. Halmos, "Finite-dimensional vector spaces", Undergraduate Texts in Mathematics, Berlin, New York: Springer-Verlag, 1974.

[8] F. Harary, "Graph Theory", Addison-Wesley, Reading, MA, 1969.

[9] S. Hoory, N. Linial, A. Widgerson, "Expander graphs and their applications", *Bulletin (New series) of the American Mathematical Society* **43** no. 4, pp. 439–561, 2006.

[10] Ph. G. H. Lehot, "An Optimal Algorithm to Detect a Line Graph and Output Its Root Graph", *J. ACM* **21** no. 4, pp. 569–575, 1974.

[11] D. Liu, S. Trajanovski, P. Van Mieghem, "ILIGRA: An Efficient Inverse Line Graph Algorithm", *Journal of Mathematical Modelling and Algorithms in Operations Research* **14** no. 1, pp. 13–33, 2015.

[12] B. Mohar, "Eigenvalues, Diameter, and Mean Distance in Graphs", *Graphs Combin.* **7**, pp. 53–64, 1991.

[13] N. D. Roussopoulos, "A max $\{m, n\}$ algorithm for determining the graph H from its line graph G", *Inform. Process. Lett.* **2** no. 4, pp. 108–112, 1973.

[14] S. Pemmaraju, S. Skiena, "Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica", Reading, MA: Addison-Wesley, 1990.

[15] K. Thulasiraman, M. N. S. Swamy, "Graphs: Theory and Algorithms", Wiley, 1992.

[16] T. W. Williams, L. M. Maxwell, "The decomposition of the graph and the introduction of the new class of subgraphs", *SIAM J. Appl. Math.* **20** no. 3, pp. 385–389, 1971.

[17] H. Whitney, "Congruent graphs and the connectivity of graphs", *American Journal of Mathematics* **54** no. 1, pp. 150–168, 1932.