Generalized Primitive Recursive Arithmetical and String Functions

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ABSTRACT
The notion of generalized primitive recursive arithmetical function was introduced in [1]and in [2] – the notion of generalized primitive recursive string function. It is proved that the string function is generalized primitive recursive if and only if its representative arithmetical function is generalized primitive recursive.

Keywords
String function, primitive recursive function, superposition, alphabetic primitive recursive.

1. INTRODUCTION

In [1], the notion of primitive recursive arithmetical function is generalized in the following way: the primitive recursive arithmetical functions are defined in ausual way, for example, as in [3]; the generalized primitive recursive arithmetical functions are defined the same way, but the arithmetical function *U(x),* defined now here, is added to the basic functions.
In [2], the notion of primitive recursive string function ([3], [4]) is generalized in the following way: the primitive recursive string functions are defined in ausual way, for example, as in ([3], [4]); the generalized primitive recursive string functions are generalized the same way, but the string function *U(P)*,defined now here, is added to the basic functions. However, in [2] it is not considered how the generalized primitive recursive arithmetical functions and the generalized primitive recursive string functions are interconnected. In this work, for each string function we consider its representative arithmetical function ([3], [4]). It is proved that each string function is generalized primitive recursive if and only if its representative arithmetical function is generalized primitive recursive.

2. GENERALIZED PRIMITIVE RECURSIVE STRING FUNCTIONS

The notion of *primitive recursive function* (or primitive recursive arithmetical function) is defined in a usual way ([3], [4], [5]).
Primitive recursive functions (*PRF*) are functions obtained from the *basic functions* 0, $I_l^k(x_1,x_2,...,x_k)=x_l$ (where $k \geq 1$, $1 \leq l \leq k$), $S(x)=x+1$, by the operators of *superposition*- $f=S(g,h_1,h_2,...,h_m)$ and *primitive recursion*- $f=R(\alpha,\beta)$ ([3], [4], [5]).
The notion of *generalized primitive recursive function* (G***PRF***) is defined similarly to the notion of *PRF* with the only difference: one-dimensional function *U*(*x*), undefined everywhere, is added to the set of basic functions.
The notion of many-dimensional primitive recursive string function is given in [3], [4].

Let *A* be an alphabet, i.e., a list of $p>1$ different symbols, $A=\{a_1,a_2,\dots,a_p\}$. By A^*we denote the set of all words in *A* (including the empty wordΛ).
We say that the function F is an *n-dimensional string function* $n \geq 1$ in the alphabet A if for any *n*-tuple $(P_1,P_2,\cdots,P_n)$, where all P_i are words in *A*, the value $F(P_1,P_2,\cdots,P_n)$ is either undefined, or is a word in *A*. By $!F(P_1,P_2,\cdots,P_n)$, we denote the statement: "the value $F(P_1,P_2,\cdots,P_n)$ is defined."
Below we consider only string functions in the fixed alphabet *A*.
Basic string functions are defined as functions of the following kinds.

1. One-dimensional function $\boldsymbol{D}(P)$ such that $\boldsymbol{D}(P)=\Lambda$ for any word P in *A*.
2. One-dimensional function $\boldsymbol{S}_i(P)$, where $1 \leq i \leq p$ such that $\boldsymbol{S}_i(P)=Pa_i$, for any word P in *A*.
3. *n*-dimensional functions $\boldsymbol{I}_m^n(P_1,P_2,\cdots,P_n)$ where $n \geq 1$, $1 \leq m \leq n$, such that $\boldsymbol{I}_m^n(P_1,P_2,\cdots,P_n)=P_m$ for any *n*-tuple $(P_1,P_2,\cdots,P_n)$ of words in *A*.

The *superposition* operator - $F=\boldsymbol{S}(G,\ G_1,\ G_2,\cdots,\ G_n)$ is definedin a natural way.
The operator $\boldsymbol{R}$ of *alphabetic primitive recursion* is defined as follows. If G is an *n*-dimensional string function, $H_1,\ H_2,\cdots,\ H_p$ are (*n*+2)-dimensional string functions, then the (*n*+1)-dimensional string function $F=\boldsymbol{R}(G,H_1,\ H_2,\cdots,\ H_p)$ is defined by the following equalities:

$$F(P_1,P_2,\cdots,P_n,\Lambda)=G(P_1,P_2,\cdots,P_n),$$
$$F(P_1,P_2,\cdots,P_n,Pa_i)=H_i\big(P_1,P_2,\cdots,P_n,P,F(P_1,P_2,\cdots,P_n,P)\big),$$

where $1 \leq i \leq p$ and $P_1,P_2,\cdots,P_n,P$ are any words in *A*.
We say that a string function is a *primitive recursive string function* (***PRSF***), if it can be obtained from the basic functions by the operators of superposition and alphabetic primitive recursion.
The notion of *generalized primitive recursive string function* (G***PRSF***) is defined similarly to the notion of ***PRSF*** with the only difference: one-dimensional function ***U***(*P*), undefined everywhere, is added to the set of basic functions.
Below the statements"F is a primitive recursive string function in the usual sense", "F is a generalized primitive recursive string function", will be denoted by $F \in \boldsymbol{PRSF}$and $F \in \boldsymbol{GPRSF}$, respectively. As it is known ([3]. [4]) every function $F \in \boldsymbol{PRSF}$is defined everywhere.
The *alphabetic number* $\pi(P)$ of the string P in A is defined by the following equalities:

$$\pi(\Lambda)=0;$$
$$\pi(a_{i_1},a_{i_2},...,a_{i_t})=i_t+i_{t-1}\cdot p+...+i_1\cdot p^{t-1}.$$

It is known (see, for example, [3]) that such enumeration of strings in A defines a one-to-one correspondence between $A^*$and the set of all natural numbers *N*={0,1,2…}. By

$\alpha_p n$ (or, shortly, αn) we denote the string in A having the alphabetic number n.

Now let us define some relations between the arithmetical functions and the string functions in a given alphabet $A=\{a_1,a_2,...,a_p\}$, where p>1. Let f be an m-dimensional arithmetical function, and F be an m-dimensional string function. We say that f is the function *representing* F (or that F is the function *representable* by f) if the following equality holds for all natural numbers $x_1,x_2...x_n$:

$$F(\alpha x_1,\alpha x_2...\alpha x_m)=\alpha f(x_1,x_2...x_m).$$

If a string function F is representable by an arithmetical function f, then we say also that the function F and f *correspond* to eachother.

Theorem 1. *For each generalized primitive recursive string function, its representative arithmetical function is a generalized primitive recursive arithmetical function.*

The proof is obtained by induction with respect to the applied string operations S and R. The representative arithmetical functions of the simplest string functions $\boldsymbol{U}(P)$, $\boldsymbol{D}(P)$, $\boldsymbol{I}_m^n$ $P_1,P_2,\cdots,P_n)$, (where $n\geq 1$, $1\leq m\leq n$); $\boldsymbol{S}_i$ $P)$, *where* $1\leq i\leq p$ are merely generalized primitive recursive functions

$\boldsymbol{U}(x)$, 0, $I_m^n(x_1,x_2,...,x_n)$, (where $n\geq 1$, $1\leq m\leq n$),

$s_i(x)=px+i$, respectively.

If the function F is obtained by superposition, $F=\boldsymbol{S}(G,\ G_1,\ G_2,\cdots,\ G_n)$,

$F(P_1,P_2,\dots,P_m)=$
$G\big(G_1\ P_1,P_2,\dots,P_m),G_2\ P_1,P_2,\dots,P_m),\dots,G_n\ P_1,P_2,\dots,P_m)\big)$

Where $G,\ G_1,\ G_2,\cdots,\ G_n$ are string functions, and $g,\ g_1,\ g_2,\cdots,\ g_n$ are the corresponding arithmetical representative functions. Representative for F will be the function

$f(x_1,x_2,\dots,x_m)=$
$g\big(g_1\ x_1,x_2,\dots,x_m),g_2\ x_1,x_2,\dots,x_m),\dots,g_n\ x_1,x_2,\dots,x_m)\big)$.

If the function F is obtained by recursion $F=\boldsymbol{R}\ G,H_1,\ H_2,\cdots,\ H_p)$, then we consider S-images of the corresponding functions (see [1], [2]).

If f is an arithmetical function, then the *S-image* of f is defined as a total function f^*such that

$$f^*\ x_1,x_2,\cdots,x_n)=\begin{cases}f(x_1,x_2,\cdots,x_n)+1, \text{when } !f(x_1,x_2,\cdots,x_n),\\ 0, \text{otherwise.}\end{cases}$$

If F is a string function, then the *S-image* of F is defined as a total function F^* such that

$$F^*\ P_1,P_2,\cdots,P_n)=\begin{cases}S_1\ F(P_1,P_2,\cdots,P_n)), \text{when } !F(P_1,P_2,\cdots,P_n),\\ \Lambda, \text{otherwise.}\end{cases}$$

In [1], it is proved that the arithmetical function f is generalized primitive recursive if and only if the S-image $f^*$is primitive recursive. In [2], it is proved that any *string function F in A is a generalized primitive recursive string function if and only if its S-image $F^*$is a primitive recursive string function in the usual sense.*

If $F=\boldsymbol{R}\ G,H_1,\ H_2,\cdots,\ H_p)$, $G,\ H_1,\ H_2,\ \cdots,\ H_p$ are generalized primitive recursive string functions, $G^*,H_1^*,H_2^*,\cdots,H_p^*$ are S-images of the functions $G,H_1,\ H_2,\cdots,\ H_p$, respectively, then the S-image F^* of the function F satisfies the following equalities:

$F^*\ P_1,P_2,\cdots,P_n,\Lambda)=$
$G^*\ P_1,P_2,\cdots,P_n),F^*\ P_1,P_2,\cdots,P_n,Pa_1)=$
$H_1^{**}\ P_1,P_2,\cdots,P_n,P,F^*\ P_1,P_2,\cdots,P_n,P)),$

$$\vdots$$

$F^*\ P_1,P_2,\cdots,P_n,Pa_p)=$
$H_p^{**}\ P_1,P_2,\cdots,P_n,P,F^*\ P_1,P_2,\cdots,P_n,P)),$

where for any i such that $1\leq i\leq p$,

$$H_i^{**}\ P_1,P_2,\cdots,P_n,P,Q)=$$
$$\Pi_2(H_i^*\ P_1,P_2,\cdots,P_n,P,Q\dot{-}a_1),Q),$$

and the primitive recursive string function Π_2 is defined in [2].

Since the string function $F^*$is defined by alphabetic primitive recursion via primitive recursive string functions, F^* is a primitive recursive string function, then representative for F^*arithmetical function $f^*$is primitive recursive, see [3]. And as the representative for the generalized primitive recursive string function F will be the arithmetical generalized primitive recursive function f defined as follows.

$f(x_1,x_2,\dots,x_n)=$
$Br\ f^*\ x_1,x_2,\dots,x_n)\dot{-}1,\overline{Sg}f^*\ x_1,x_2,\dots,x_n)).$

Definition of GPRF function Br (x, y), see in [1].

By Br (x, y) ("Branching function") we denote a $GPRF$ defined as the operation of primitive recursion:

$$Br(x,0)=x;$$
$$Br(x,S(y))=U(I_1^3\ x,y,Br(x,y))).$$

So, $Br(x,0)=x$ when $y=0$, and $Br(x,y)$ is undefined when $y>0$.

The theorem is proved.

Theorem 2. *Each generalized primitive recursive arithmetical function is the representative arithmetical function of some generalized primitive recursive string function.*

The theorem is proved with the help of three lemmas.

Let us introduce a function ν such that for every natural number n (cf. [3]):

$$\nu(n)=a_1^n=\underbrace{a_1a_1...a_1}_{n\ times};$$
$$\nu(0)=a_1^0=\Lambda,$$

where a_1 is the first letter in A.

Lemma 1. (cf. [3],p. 216, Lemma 1) *For each generalized primitive recursive arithmetical function $f(x_1,\dots,x_n)$ there exists a generalized primitive recursive string function $F(P_1,\dots,P_n)$ satisfying the identity*

$$F\big(\nu\, x_1,\dots,\nu\, x_n\big)=\nu\, f(x_1,\dots,x_n).$$

The proof is given by induction with respect to the number of applied operations S and R.

<u>**Lemma 2**</u>. *There exists a primitive recursive string function T such that $T\big(\nu(m)\big)=\alpha m$ for every natural number m.*

The proof is given in [3] (see [3], p. 217).

<u>**Lemma 3**</u>. *The one-dimensional string function $\gamma(Q)=\nu(\pi(Q))$ is primitive recursive*.

The proof is given in [3] (see [3], p. 217).

The proof of the theorem is obtained in the following way.

Suppose $f(x_1,\dots,x_n)$ is a generalized primitive recursive arithmetical function, and the string function G is defined as follows:

$$G(P_1,\dots,P_n)=\alpha f\left(\pi\ \ P_1),\dots,\pi\ \ P_n)\right).$$

By Lemma 2, we have

$$\alpha f\left(\pi\ \ P_1),\dots,\pi\ \ P_n)\right)=T(\nu f\left(\pi\ \ P_1),\dots,\pi\ \ P_n)\right)$$

By Lemma 1, there exists a generalized primitive recursive string function F such that

$\nu f\left(\pi\ \ P_1),\dots,\pi\ \ P_n)\right)=F\left(\nu\pi\ \ P_1),\dots,\nu\pi\ \ P_n)\right)=$
$F\big(\gamma(P_1),\dots,\gamma(P_n)\big).$

Thus, we have

$$G(P_1,\dots,P_n)=T\left(F\big(\gamma(P_1),\dots,\gamma(P_n)\big)\right).$$

Since all the three functions T, F, γ are generalized primitive recursive string functions then the function G is also a generalized primitive recursive string function.

The theorem is proved.

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